QES calculation

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Let $2\Phi_H(\partial I) + S_{\text{vN}}(R \cup I) = \frac{c}{24} f(\partial I)$ then $S(R) = \min \text{ext}_I \left[\frac{c}{48} f(\partial I)\right]$.

$$\frac{c}{24} f(\partial I) = \frac{c}{24}\left[1 + \frac{2\Phi_s}{c}\left(\frac{1 + x^+_{\partial I} x^-_{\partial I}}{1 - x^+_{\partial I} x^-_{\partial I}}\right) - \left(\frac{1 + x^+_{\partial I} x^-_{\partial I}}{1 - x^+_{\partial I} x^-_{\partial I}}\right) \log\left(x^+_{\partial I}\right)\right]$$
$$+ \frac{c}{12} \log\left[\frac{4\left(x^+_{\partial I} x^+_Q\right)^{1/2}\left(x^-_{\partial I} - x^-_Q\right)^2}{\epsilon^2\left(1 - x^+_{\partial I} x^-_{\partial I}\right)\left(1 - x^+_Q x^-_Q\right)}\right]$$
$$+ \frac{c}{12} \log\left[\frac{4\left(x^+_{\partial I} x^+_Q\right)^{1/2} \log\left(x^+_Q / x^+_{\partial I}\right)^2}{\epsilon^2\left(1 - x^+_{\partial I} x^-_{\partial I}\right)\left(1 - x^+_Q x^-_Q\right)}\right]$$

Now to find the minima of $f(\partial I)$ we partial differentiate it with respect to $x^+_{\partial I}$ and $x^-_{\partial I}$.

Partial differentiation with respect to $x^+_{\partial I}$ gives

$$\frac{\partial}{\partial x^+_{\partial I}} f(\partial I) = \frac{\partial}{\partial x^+_{\partial I}}\left(\left[1 + \frac{2\Phi_s}{c}\left(\frac{1 + x^+_{\partial I} x^-_{\partial I}}{1 - x^+_{\partial I} x^-_{\partial I}}\right) - \left(\frac{1 + x^+_{\partial I} x^-_{\partial I}}{1 - x^+_{\partial I} x^-_{\partial I}}\right) \log\left(x^+_{\partial I}\right)\right]\right.$$
$$\Longrightarrow \qquad + 2 \log\left[\frac{4\left(x^+_{\partial I} x^+_Q\right)^{1/2}\left(x^-_{\partial I} - x^-_Q\right)^2}{\epsilon^2\left(1 - x^+_{\partial I} x^-_{\partial I}\right)\left(1 - x^+_Q x^-_Q\right)}\right]$$
$$\left. + 2 \log\left[\frac{4\left(x^+_{\partial I} x^+_Q\right)^{1/2} \log\left(x^+_Q / x^+_{\partial I}\right)^2}{\epsilon^2\left(1 - x^+_{\partial I} x^-_{\partial I}\right)\left(1 - x^+_Q x^-_Q\right)}\right]\right)$$

$$0 = \left[\frac{2\Phi_s}{c}\left(\frac{2x^-_{\partial I}}{(1 - x^+_{\partial I} x^-_{\partial I})^2}\right) - \left(\frac{2x^-_{\partial I}}{(1 - x^+_{\partial I} x^-_{\partial I})^2}\right) \log\left(x^+_{\partial I}\right) - \left(\frac{1 + x^+_{\partial I} x^-_{\partial I}}{1 - x^+_{\partial I} x^-_{\partial I}}\right) \frac{1}{x^+_{\partial I}}\right]$$
$$\Longrightarrow \qquad + \frac{\partial}{\partial x^+_{\partial I}}\left(2 \log\left[\frac{\left(x^+_{\partial I}\right)^{1/2}}{\left(1 - x^+_{\partial I} x^-_{\partial I}\right)}\right]\right.$$
$$+ 2 \log\left[\frac{\left(x^+_{\partial I}\right)^{1/2} \log\left(x^+_Q / x^+_{\partial I}\right)^2}{\left(1 - x^+_{\partial I} x^-_{\partial I}\right)}\right]$$

$$
\boxed{\begin{aligned}
0 &= \left[\frac{2\Phi_s}{c}\left(\frac{2x^-_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})^2}\right) - \left(\frac{2x^-_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})^2}\right)\log\left(x^+_{\partial I}\right) - \left(\frac{1+x^+_{\partial I}x^-_{\partial I}}{1-x^+_{\partial I}x^-_{\partial I}}\right)\frac{1}{x^+_{\partial I}}\right] \\
\implies \quad &+ 2\left[\frac{1}{2x^+_{\partial I}} + \frac{x^-_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})}\right] \\
&+ 2\left[\frac{1}{2x^+_{\partial I}} + \frac{x^-_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})} - \frac{2}{x^+_{\partial I}\log\left(x^+_Q/x^+_{\partial I}\right)}\right]
\end{aligned}}
$$

Similarly with respect to $x^-_{\partial I}$ gives

$$
\begin{aligned}
\frac{\partial}{\partial x^-_{\partial I}} f(\partial I) = \frac{\partial}{\partial x^-_{\partial I}}\Bigg(&\left[1+\frac{2\Phi_s}{c}\left(\frac{1+x^+_{\partial I}x^-_{\partial I}}{1-x^+_{\partial I}x^-_{\partial I}}\right) - \left(\frac{1+x^+_{\partial I}x^-_{\partial I}}{1-x^+_{\partial I}x^-_{\partial I}}\right)\log\left(x^+_{\partial I}\right)\right] \\
&+ 2\log\left[\frac{4\left(x^+_{\partial I}x^+_Q\right)^{1/2}\left(x^-_{\partial I}-x^-_Q\right)^2}{\epsilon^2\left(1-x^+_{\partial I}x^-_{\partial I}\right)\left(1-x^+_Qx^-_Q\right)}\right] \\
&+ 2\log\left[\frac{4\left(x^+_{\partial I}x^+_Q\right)^{1/2}\log\left(x^+_Q/x^+_{\partial I}\right)^2}{\epsilon^2\left(1-x^+_{\partial I}x^-_{\partial I}\right)\left(1-x^+_Qx^-_Q\right)}\right]\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\partial}{\partial x^-_{\partial I}}\Bigg(\left[\frac{2\Phi_s}{c}\left(\frac{1+x^+_{\partial I}x^-_{\partial I}}{1-x^+_{\partial I}x^-_{\partial I}}\right) - \left(\frac{1+x^+_{\partial I}x^-_{\partial I}}{1-x^+_{\partial I}x^-_{\partial I}}\right)\log\left(x^+_{\partial I}\right)\right] \\
\implies \quad &+ 2\log\left[\frac{\left(x^-_{\partial I}-x^-_Q\right)^2}{(1-x^+_{\partial I}x^-_{\partial I})}\right] \\
&+ 2\log\left[\frac{1}{(1-x^+_{\partial I}x^-_{\partial I})}\right]\Bigg)
\end{aligned}
$$

$$
\boxed{\begin{aligned}
0 &= \left(\frac{2x^+_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})^2}\right)\left(\frac{2\Phi_s}{c} - \log\left(x^+_{\partial I}\right)\right) \\
\implies \quad &+ 2\left[2\frac{1}{\left(x^-_{\partial I}-x^-_Q\right)} + \frac{x^+_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})}\right] \\
&+ 2\frac{x^+_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})}
\end{aligned}}
$$

Now in the above equation if we assume $x^-_{\partial I} << 1$

$$
\begin{aligned}
& 0 = 2x^+_{\partial I}\left(\frac{2\Phi_s}{c} - \log\left(x^+_{\partial I}\right)\right) \\
\Longrightarrow \quad & + 2\left[-2\frac{1}{x^-_Q} + x^+_{\partial I}\right] \\
& + 2x^+_{\partial I}
\end{aligned}
$$

then

$$
\begin{aligned}
2x^+_{\partial I}\left(-\frac{2\Phi_s}{c} + \log\left(x^+_{\partial I}\right)\right) &= 2\left[-2\frac{1}{x^-_Q} + x^+_{\partial I}\right] + 2x^+_{\partial I} \\
\log\left(x^+_{\partial I}\right) &= 2\left[-\frac{1}{x^+_{\partial I}x^-_Q} + 1\right] + \frac{2\Phi_s}{c} \\
\Longrightarrow \qquad x^+_{\partial I} &= e^{2\left[-\frac{1}{x^+_{\partial I}x^-_Q}+1\right]+\frac{2\Phi_s}{c}} \\
\frac{-2}{x^-_Q}e^{-2-\frac{2\Phi_s}{c}} &= \frac{-2}{x^+_{\partial I}x^-_Q}e^{\left[-\frac{2}{x^+_{\partial I}x^-_Q}\right]} \\
W_0\left(\frac{-2}{x^-_Q}e^{-2-\frac{2\Phi_s}{c}}\right) &= \frac{-2}{x^+_{\partial I}x^-_Q}
\end{aligned}
$$

$$
x^+_{\partial I} = \frac{-2}{x^-_Q W_0\left(\frac{-2}{x^-_Q}e^{-2-\frac{2\Phi_s}{c}}\right)}
$$

If we instead neglect the 3rd line (2nd term in von Neumann entropy) at the beginning

$$
\begin{aligned}
& 0 = 2x^+_{\partial I}\left(\frac{2\Phi_s}{c} - \log\left(x^+_{\partial I}\right)\right) \\
\Longrightarrow \quad & + 2\left[-2\frac{1}{x^-_Q} + x^+_{\partial I}\right]
\end{aligned}
$$

then

$$
\begin{aligned}
2x^+_{\partial I}\left(-\frac{2\Phi_s}{c}+\log\left(x^+_{\partial I}\right)\right) &= 2\left[-2\frac{1}{x^-_Q}+x^+_{\partial I}\right] \\
\log\left(x^+_{\partial I}\right) &= \left[-\frac{2}{x^+_{\partial I}x^-_Q}+1\right]+\frac{2\Phi_s}{c} \\
\implies \qquad x^+_{\partial I} &= e^{-\frac{2}{x^+_{\partial I}x^-_Q}+1+\frac{2\Phi_s}{c}} \\
\frac{-2}{x^-_Q}e^{-1-\frac{2\Phi_s}{c}} &= \frac{-2}{x^+_{\partial I}x^-_Q}e^{\left[-\frac{2}{x^+_{\partial I}x^-_Q}\right]} \\
W_0\left(\frac{-2}{x^-_Q}e^{-1-\frac{2\Phi_s}{c}}\right) &= -\frac{2}{x^+_{\partial I}x^-_Q}
\end{aligned}
$$

Now if we take the 1st boxed equation and assume $x^-_{\partial I} << 1$

$$
\boxed{
\begin{aligned}
0 = &\left[\frac{2\Phi_s}{c}\left(\frac{2x^-_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})^2}\right)-\left(\frac{2x^-_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})^2}\right)\log\left(x^+_{\partial I}\right)-\left(\frac{1+x^+_{\partial I}x^-_{\partial I}}{1-x^+_{\partial I}x^-_{\partial I}}\right)\frac{1}{x^+_{\partial I}}\right] \\
\implies \quad &+2\left[\frac{1}{2x^+_{\partial I}}+\frac{x^-_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})}\right] \\
&+2\left[\frac{1}{2x^+_{\partial I}}+\frac{x^-_{\partial I}}{(1-x^+_{\partial I}x^-_{\partial I})}-\frac{2}{x^+_{\partial I}\log\left(x^+_Q/x^+_{\partial I}\right)}\right]
\end{aligned}
}
$$

$$
\begin{aligned}
0 = &\left[\left(\frac{2\Phi_s}{c}-\log\left(x^+_{\partial I}\right)\right)2x^-_{\partial I}-\frac{1}{x^+_{\partial I}}\right] \\
\implies \quad &+2\left[\frac{1}{2x^+_{\partial I}}+x^-_{\partial I}\right] \\
&+2\left[\frac{1}{2x^+_{\partial I}}+x^-_{\partial I}-\frac{2}{x^+_{\partial I}\log\left(x^+_Q/x^+_{\partial I}\right)}\right]
\end{aligned}
$$

This gives us the approximation

$$
x^-_{\partial I} \approx \frac{\frac{1}{x^+_{\partial I}}-\frac{2}{x^+_{\partial I}\log\left(x^+_{\partial I}\right)}}{\log\left(x^+_{\partial I}\right)-\frac{2\Phi_s}{c}-4}
$$

at late times

$$
x^-_{\partial I} \approx \frac{1}{x^+_{\partial I}\log\left(x^+_{\partial I}\right)}
$$