# Quantum covariant bit threads

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Based on 2 upcoming papers with Matthew Headrick & Andrew Rolph
'Quantum bit threads 2' & 'Quantum covariant bit threads'
Slides at www.ksr.onl/slides

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# Intro

It is generally hard to compute Entanglement Entropy $S_{vNE} = -\operatorname{tr}(\rho \ln \rho)$ for QFTs. But for gapped large-$N$ CFTs, we have a semiclassical holographic description, making things easier.

Recall the main result of Black Hole Thermodynamics (QFT in curved spacetime):

$$S_{BH} = \frac{c^3 A}{4G\hbar} = \frac{A}{4}$$

1. $S_{BH}$ is often called the "classical" entropy of a black hole but strictly speaking the proportionality constant diverges as $\hbar \to 0$.

This diverging quantity will always increases due to the second law of thermodynamics since the outside entropy is negligible. This was rigorously proved by Hawking. Even when black hole energy or mass decreases like in Penrose process, the area increases.

# Killing horizons are extremal surfaces

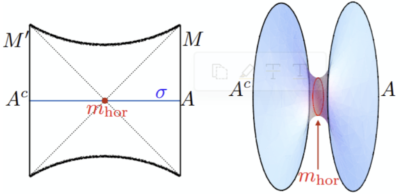



Figure: Left: Penrose diagram of two-sided static asymptotically AdS black hole. Right: The induced wormhole geometry on the Cauchy slice $\sigma$.

Note that BH event horizons are also **extremal surfaces** in $t = const$ Cauchy slices. This can be generalized to other Killing horizons, such as the Hubble sphere $r = \sqrt{3/\Lambda}$ for de Sitter space. Note the **sign flip** between $r$, $t$ for both Killing horizons.

$$ds^2 = \left(1 - \frac{\Lambda}{3} r^2\right) dt^2 - \frac{1}{\left(1 - \frac{\Lambda}{3} r^2\right)} dr^2 - r^2 d\Omega^2$$

# Generalized entropy

If matter is quantum instead of classical then there will be Hawking radiation corresponding to $T_{\mathrm{H}} = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}$. $A$ no longer increases but $S_{\mathrm{gen}}$ still increases.

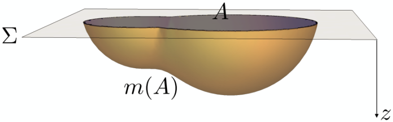
$$S_{\mathrm{gen}} = \frac{\mathrm{A}}{4G_N} + S_{\mathrm{out}}$$

1. The area term is $O(\hbar^{-1})$ but once we add $S_{out}$ its accurate up to all orders of $\hbar$.
2. UV finiteness: Most of the entanglement is short-ranged, in the lattice approximation nearby lattice sites so there is a generic "area-law" divergence $\frac{A}{\epsilon_{UV}^2}$ in $S_{\mathrm{out}}$. The "infinite renormalization of $G_N$": $\frac{1}{4G_N} \to \frac{1}{4G_N} - \frac{1}{\epsilon_{UV}^2}$ cancels that out making the formula UV finite. **Note**: Gravity is **classical**. $\frac{1}{4G_N}$ changes due to the back reaction of matter $UV$ divergences. $\epsilon_{UV}$ is matter UV cutoff.

# RT formula (Ryu–Takayanagi 2006)

HEE is a generalization of black hole thermodynamics but instead of just Killing horizons we **assign** entropy to all extremal surfaces.

$$S(A) = \min_{\gamma} \frac{\text{Area}(\gamma)}{4G_N}$$




Its not necessary for spacetime to be **static**. A Cauchy slice $\sigma$ with time reflection symmetry is enough. $\gamma$ is any surface on this Cauchy slice that is homologous to $A$. We can interpret $S_{BH}$ as HEE such that in that case $\partial A$ and $\partial A^c$ are null sets.

# RT properties

**Main difference**: RT= fine grained (microstate) vs $S_{BH}$= coarse grained (macrostate). 2nd law of thermodynamics = total entropy (ignorance about the universe) keeps increasing but not if you already know every microscopic detail in which case it remains constant under unitary time evolution. In the 2 sided static black hole both are the same.

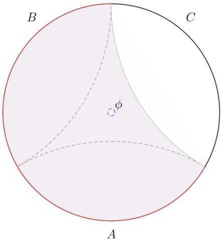



Figure: Subregion duality and error correction

# RT properties

**Strong subadditivity**:

$$S(AB) + S(BC) \geq S(B) + S(ABC)$$

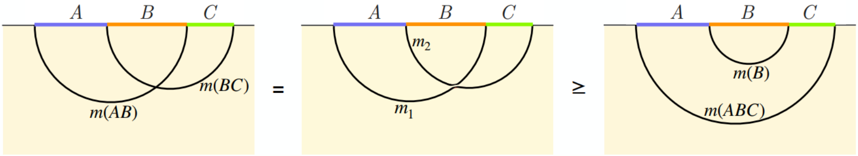



**Figure 18**: Illustration of proof that RT obeys the strong subadditivity inequality (5.52). The two surfaces $m(AB)$, $m(BC)$ are cut and pasted to give surfaces $m_1$, $m_2$ with the same total area. $m_1$ and $m_2$ are homologous to $ABC$ and $B$, via the regions $r(AB) \cup r(BC)$ and $r(AB) \cap r(BC)$, so their areas bound $S(ABC)$ and $S(B)$, respectively.

# Multiple extremal surfaces

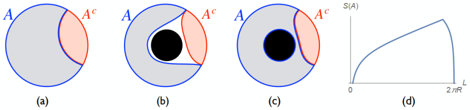
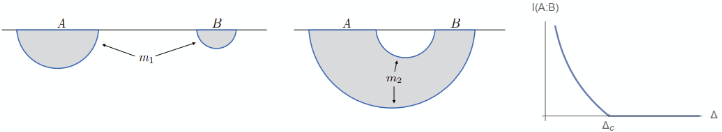



**Figure 13**: RT surfaces $m(A)$, $m(A^c)$ and homology regions $r(A)$, $r(A^c)$ for an interval $A$ and the complementary interval $A^c$ on a circle. In the vacuum or other pure state, shown in (a), $m(A)$ and $m(A^c)$ coincide (more precisely $m(A^c) = -m(A)$, although orientations are not shown on the diagram). In a black hole geometry, there are two possible configurations for $m(A)$, neither of which equals $m(A^c)$: a single geodesic which is homotopic to $A$ (b), or the union of the horizon and $m(A^c)$ (c). $S(A)$, plotted in (d), is given by the smaller of the areas of the surfaces shown in (b) and (c) ((c) when $A$ covers almost the entire circle, otherwise (b)).



# Covariant HRT formula (Hubeny Rangamani Takayanagi 2007)

We need to find extremal spacelike surface homologous to $A$.

$$S = \underset{\gamma}{\text{ext}} \left[ \frac{\text{Area}(\gamma)}{4G_N} \right]$$

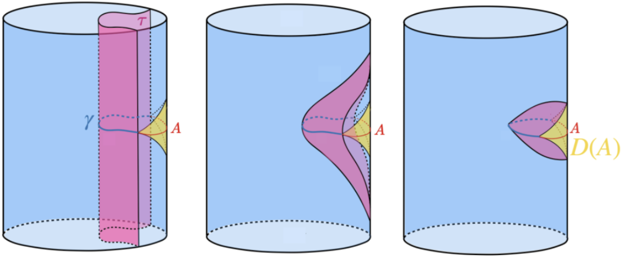
Maximin prescription (Wall 2012)

$$S_{-}(A:B) := \sup_{\sigma \in \mathcal{S}} \inf_{\gamma \in \Gamma_\sigma} \left[ \frac{\text{Area}(\gamma)}{4G_N} \right]$$

Minimax prescription (Headrick Hubeny 2022)

$$S_{+} = \inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \mathcal{S}} \left[ \frac{\text{Area}(\sigma \cap \tau)}{4G_N} \right]$$

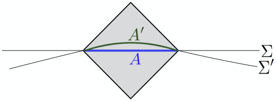
# Time sheets: natural alternatives to Cauchy slices

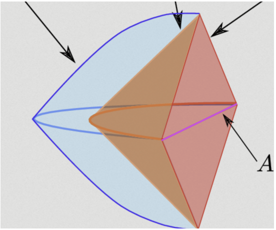



Just like Cauchy slices must contain $A$ in their boundary, the time sheets we consider must touch $\partial A$ but not necessarily $\partial D(A)$. A time-sheet $\tau$ is a piecewise-timelike hypersurface.

# Causal structure

Covariant HRT formula has causal structure encoded in it. On both sides we should now attribute the entropy to $D(A)$ the domain of dependence of $A$ and domain of dependence of $r_{\gamma_{HRT}}$ (entanglement wedge $\mathcal{W}(A)$). In the below the causal wedge is contained inside $\mathcal{W}(A)$.




**Subregion duality**: Just like HEE is a vast generalization of $S_{BH}$, AdS/CFT is a vast generalization of HEE. HEE just says that EE of $\rho_A$ maps to an extremal bulk surface. AdS/CFT says the whole $\rho_A$ maps to $\mathcal{W}(A)$.

# QES formula (Engelhardt Wall 2014)

We can generalize $S_{gen}$ from Killing horizons to “quantum” extremal surfaces. Gravity = classical and Matter = Quantum.

$$S_{gen} = \min_{\gamma} \left[ \frac{\text{Area}(\gamma)}{4G_N} + S_{\text{bulk}}(r_\gamma) \right]$$

1. UV finite: Same as black hole.
2. IR divergences: UV divergences in CFT match with the IR divergence in AdS. To get IR finite quantities we need to always use the mutual information $I(A:B) = S(A) + S(B) - S(AB)$. See fig 9.

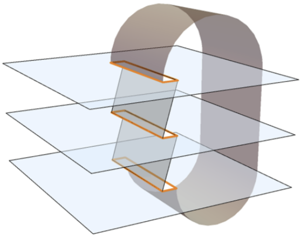
# Generalisations of HEE

1. Beyond *AdS*: Bousso Penington 2208.04993, Witten et al 2206.10780 von Neumann algebra approach, Pseudo and timelike entropy by Takayanagi et al 2302.11695, Shaghoulian Susskind 2201.03603 etc.
2. Pennington et al 2501.08308 proposed a generalization when gravity is quantum. They only studied the first-order correction $\hat{h}_{\mu\nu}$. Area is promoted to an operator.

All these QES generalizations are very speculative. There are other related speculative proposals like Complexity=Volume or Action etc. But unlike these RT/HRT/QES are on proven from $Z_{AdS} = Z_{CFT}$ and are on firm footing.

# Proof (Original: Lewkowycz Maldacena Simpler: Xi Dong)

Replica trick: $n \to 1^+$ for $S_A^n = \frac{1}{1-n} \log \mathrm{Tr}\, \rho_A^n$. But we need to use $\tilde{S}_A^n = S_A^n + n(n-1)\partial_n S_A^n$ as they have better holographic interpretation.



To calculate $\mathrm{Tr} \rho_A^n$ we need to take $n$ copies of the manifold and cut the region $A$ in each and then glue those copies together as shown and calulcate the path integral. It gives that the Modular Rényi entropies are just the area of codimension 2 cosmic branes with tension $T = \frac{n-1}{4nG_N}$. This tension back reacts. Example: In $AdS_3/CFT_2$, its a massive particle whose spacelike geodesic we need to find without assuming test mass limit.

# Classical Static Bit Threads (Freedman Headrick 2016)

**Basic idea**: We consider all possible flows in the spacetime that have a norm bound and try to maximize the flux on the boundary subregion $A$ then this maximum possible flux is just $\frac{A}{4}$ or entanglement entropy.

$$\nabla_\mu v^\mu = 0\,, \qquad |v| \leq \frac{1}{4G_N}\,.$$

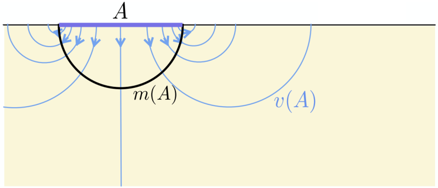
$$\int_\gamma v := \int_\gamma \sqrt{h}\, n_\mu v^\mu\,,$$

$$\Rightarrow n_\mu v^\mu \leq \frac{1}{4G_N} \Rightarrow \int_\gamma v \leq \frac{\text{Area}(\gamma)}{4G_N}.$$

$$\int_A v = \int_\gamma v \leq \frac{\text{Area}(\gamma)}{4G_N}$$

$$\sup_v \int_A v \leq \inf_\gamma \frac{\text{Area}(\gamma)}{4G_N}$$

This inequality is always saturated due to the **max-flow min-cut theorem** in optimization theory. The main difference between the popular discrete graphs case and this is that our spacetime is continuous.
**Intuition**: If water is going on a pipe then the maximum flux is decided by the minimal surface in the pipe.




**Optimal flows**: Highly non unique. The only thing we can say about the flux is its values on $\gamma_{RT}$. On $\gamma_{RT}$ it saturates the norm bound and is in the direction of $n$. Except $\gamma_{RT}$, at the remaining places it has a lot of freedom. We can always chose an optimal flow bit thread set such that during a phase transition they don't jump and vary continuously unlike minimal surfaces which jump.

# Quantum Static Bit Threads (Rolph 2021)

**Intuition**: Each bit thread (integral curves of the flow) is a single qubit worth of entanglement between $A$ and $\bar{A}$. But in that case, there was no entanglement between the 2 bulk regions. Once we introduce entanglement between the bulk regions, **we need to allow for bit threads to start or end on the bulk**. In the original case, bit threads started or ended only on the boundary, as entanglement was **only present in the boundary theory** and not in bulk theory. ER=EPR conjecture gives another interpretation where such a bit thread can be considered jumping via a Planck-sized wormhole.

# Quantum Static Bit Threads

The QSBT prescription is

$$S(A) = \max_{v} \int_A n_\mu v^\mu$$

with $v$ subject to the constraints

$$|v| \leq \frac{1}{4G_N} \text{ and } \forall \gamma : \left( -\int_{r_\gamma} \nabla_\mu v^\mu(x) \leq S_{bulk}(r_\gamma) \right)$$

Number of "**negative charges**" is bounded above. **Optimal flows**:
1) As before the norm bound is saturated on the $\gamma_{QES}$.
2) The divergence bound is saturated for the bulk homology region between $\gamma_{QES}$ and $A$. That is

$$-\int_{r(\gamma_{QES})} \nabla_\mu v^\mu(x) = S_{bulk}(r(\gamma_{QES})).$$

3) Away from $\gamma_{QES}$ there is a lot of freedom for the optimal flow.

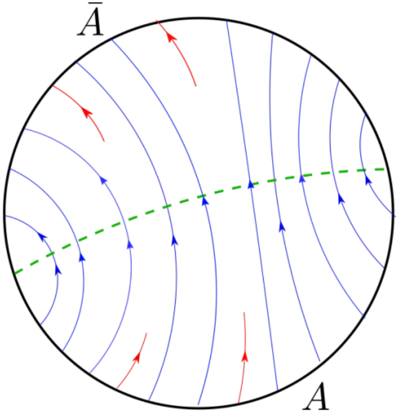
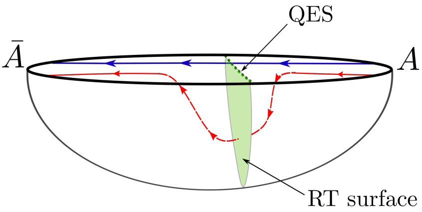



Figure: An optimal flow for QSBT prescription. The blue lines are ordinary bit threads. The red lines are due to having non zero divergence for the flow. The number of red lines measures the bulk entanglement between the 2 bulk regions.

# Double holography interpretation

The red threads are going from $r(A)$ to $r(A^c)$ and count $S_{bulk}$ in the higher-dimensional bulk.



# Classical Covariant Bit Threads (Headrick Hubeny 2022)

Relax the assumption of time reflection symmetry or static spacetime. So, we are not confining the flow to a single Cauchy slice.

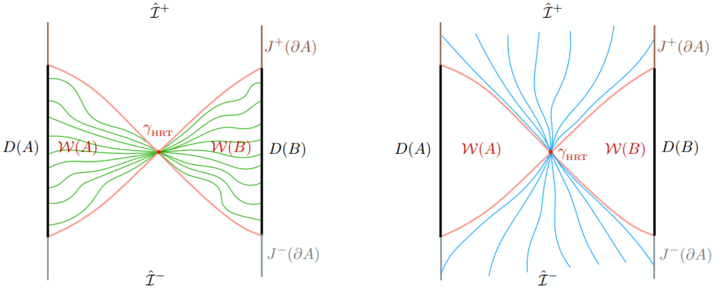



Figure: An optimal $V$-flow and $U$-flow respectively. $V$-flows are only within the 2 entanglement wedges $\mathcal{W}(A)$ and $\mathcal{W}(B)$. Similarly, $U$-flows are only outside of the 2 entanglement wedges. Both of these flows naturally discover the entanglement wedges.

With area-based formulas, we can't know if it's the causal wedge or the entanglement wedge, and originally, people thought the dual of $D(A)$ was the causal wedge. But bit threads naturally show the entanglement wedge.

# Classical Covariant Bit Threads

The CCBT prescription is

$$S(A) = \sup_V \int_{D(A)} *V \, .$$

where $V$ is a 1-form subject to $(d * V = 0)$ and one of the two equivalent norm bounds below
1. $V = 0$ in the bulk chronal future and past of $\partial A$ and for every bulk timelike curve $\int dt \, |V_\perp| \leq \frac{1}{4G_N}$ where $t$ is the proper time along the curve and $V_\perp$ is the projection of $V$ perpendicular to the curve.
2. There exists a function $\phi$ in the bulk that equals $\pm 1/2$ on $\hat{\mathcal{I}}^\pm \cup J^\pm(\partial A)$ , such that the 1-forms $d\phi \pm 4G_N V$ are every where future-directed causal.

Similarly we can define $U$ flows $S(A) = \inf_U \int_\sigma *U$ with norm bound for every spacelike bulk curve connecting $D(A)$ to $D(B)$, $\int ds \, |U_\perp| \geq 1$ .

# Threads: Integrated flows

Maximize $\mu(\mathcal{P})$ over measure $\mu$ on $\mathcal{P}$ subject to: $\forall\, q \in \mathcal{Q}, \quad \int_{\mathcal{P}} d\mu(p)\, \Delta(q,p) \leq 1$

$$L[\mu,\nu] = \mu(\mathcal{P}) + \int_{\mathcal{Q}} d\nu \left(1 - \int_{\mathcal{P}} d\mu(p)\, \Delta(q,p)\right)$$

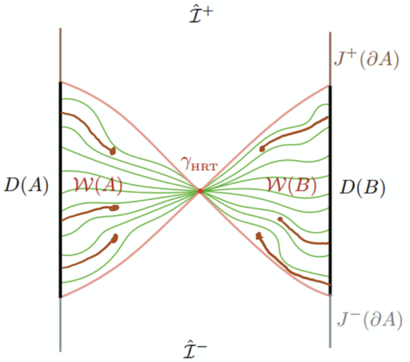
$$= \int_{\mathcal{Q}} d\nu + \mu(\mathcal{P}) - \int_{\mathcal{Q}} d\nu \left(\int_{\mathcal{P}} d\mu(p)\, \Delta(q,p)\right)$$

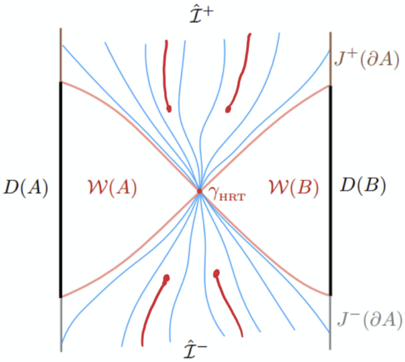
$$= \nu(\mathcal{Q}) + \int_{\mathcal{P}} d\mu(p) - \int_{\mathcal{Q}} d\nu \left(\int_{\mathcal{P}} d\mu(p)\, \Delta(q,p)\right)$$

$$= \nu(\mathcal{Q}) + \int_{\mathcal{P}} d\mu(p) \left(1 - \int_{\mathcal{Q}} d\nu(q) \Delta(q,p)\right)$$

$$\text{Minimize}\, \nu(\mathcal{Q}) \quad \forall p \in \mathcal{P}, 1 - \int_{\mathcal{Q}} d\nu(q) \Delta(q,p) \leq 0$$

# Quantum Covariant Bit Threads



# Quantum Covariant Bit Threads

The only change we need to do is replace the zero divergence condition with

$$\forall\gamma : \left( -\int_{D(r_\gamma)} d * V \leq S_{bulk}(D(r_\gamma)) \right)$$

$$\forall\gamma : \left( \int_{\mathcal{I}^-(\gamma)} d * U \geq S_{bulk}(D(r_\gamma)) \right)$$

**Classical limit**: $\hbar \to 0$ implies we need to drop the $S_{bulk}$ term.
**Static limit**: Confined to a single Cauchy surface. We can replace $\int dt\,|V_\perp| \leq \frac{1}{4G_N}$ with $|V| \leq \frac{1}{4G_N}$.

# Optimal solution

1. All the flux passes through $\gamma_{QES}$, and the flux on the remaining $\mathcal{H}(A)$ is 0. On the $\gamma_{QES}$, it will be a Dirac delta function. So, it will be of the form $V = \frac{1}{4G_N}\delta(t - t_c)n$ where $n$ is some unit normal direction that is a function of the points on $\gamma_{QES}$.
2. The divergence bound is saturated (so the number of places bit threads randomly appear is fixed from this)

$$-\int_{D(\gamma_{QES})} d * V = S_{bulk}(D(\gamma_{QES}))$$

3. Bit threads can not appear outside the 2 entanglement wedges due to the norm bound.

# Thread Dualization

Let us start with the definition of $V$ threads

$$\text{Maximize } \int_{\mathcal{P}} d\mu$$
$$\forall q \in \mathcal{Q}, 1 - \int_{\mathcal{P}} d\mu(p)\Delta(q,p) \geq 0 \tag{3.1}$$
$$\forall \tau, \sigma \text{ such that } \sigma \cap \tau \sim A : \int_{\mathcal{P}} d\mu(p) N(\tau, p) \leq S_{bulk}(D(\tau \cap \sigma))$$

The corresponding flow obtained will have the following tighter constraint, but these are equivalent programs that give the same optimal values. $D(r_\gamma)$ is a light-like limiting case of time-sheets.

$$\forall \tau, \sigma : \int_{\text{Side of } \tau \text{ that has } D(A)} d * V + S_{\text{bulk}}(\sigma \cap \tau) \geq 0 \tag{3.2}$$

# $U'$ threads

$$L[\mu,\nu,\rho] = \mu(\mathcal{P}) + \int_{\mathcal{Q}} d\nu \left(1 - 4G_N \int_{\mathcal{P}} d\mu(p)\,\Delta(q,p)\right) + \int d\rho d\kappa \left(S_{bulk}(D(\tau\cap\sigma)) - \int_{\mathcal{P}} d\mu(p) N(\tau,p)\right)$$
$$= \nu(\mathcal{Q}) + \mu(\mathcal{P}) - 4G_N \int_{\mathcal{Q}} d\nu \left(\int_{\mathcal{P}} d\mu(p)\,\Delta(q,p)\right) + \int d\rho d\kappa \left(S_{bulk}(D(\tau\cap\sigma)) - \int_{\mathcal{P}} d\mu(p) N(\tau,p)\right)$$
$$= \nu(\mathcal{Q}) + \int d\rho d\kappa S_{bulk}(D(\tau\cap\sigma)) + \int_{\mathcal{P}} d\mu(p) \left(1 - 4G_N \int_{\mathcal{Q}} d\nu\,(\Delta(q,p)) - \int d\rho d\kappa N(\tau,p)\right)$$

This gives us the below $U'$ threads after optimizing over $\mu(p)$, which is essentially like a Lagrange multiplier for inequalities. $\mu$ corresponding to V threads is concave, so we maximize over it, and $\nu$ corresponding to U threads is convex, so we minimize. Similarly, $d\kappa(\sigma)$ is convex (minimize) and $d\rho(\tau)$ is concave (maximize). $d\kappa(\sigma)$ and $d\rho(\tau)$ have total value as 1, so they are interpreted as **averages** over $\sigma$ and $\tau$.

$$S(A) = \inf_{\nu(\mathcal{P})} \inf_{\kappa(\sigma)} \sup_{\rho(\tau)} \left[\nu(\mathcal{Q}) + \int d\rho d\kappa S_{bulk}(D(\tau\cap\sigma))\right]$$
$$\forall p \in \mathcal{P}, 1 - 4G_N \int_{\mathcal{Q}} d\nu(Q)\Delta(q,p) - \int d\rho d\kappa N(\tau,p) \leq 0$$

# $V'$ threads

Similarly, if we start with quantum $U$ threads and find the dual the norm bound spacelike curves without endpoints **become** $V'$ threads without endpoints and the $U$ threads become timelike constraint curves with endpoints.

$$\begin{gathered} S(A) = \sup_{\mu(\mathcal{Q})} \inf_{\kappa(\sigma)} \sup_{\rho(\tau)} \left[ \mu(\mathcal{Q}) + \int d\rho d\kappa S_{bulk} \right] \\ \forall q \in \mathcal{Q}, \quad 1 - 4G_N \int_{\mathcal{P}} d\mu(p)\Delta(q,p) - \int d\rho d\kappa N(I^+(\sigma), q) \geq 0 \end{gathered} \tag{3.3}$$

Similarly, we can get the above $V'$ threads. So, we have demarcated the classical area part and the $S_{bulk}$ part. We now have classical threads without divergence (endpoints) that have a constraint over all timelike or spacelike curves. that Classically, $V'$ and $U'$ threads become the ordinary $V$ and $U$ threads.

# $U'$ flows and $V'$ flows

By taking the tangents, we can get the corresponding flows.

$$S(A) = \inf_{U'} \inf_{\kappa(\sigma)} \sup_{\rho(\tau)} \left[ \int_{\mathcal{I}^+} *U' + \int d\rho d\kappa S_{bulk}(D(\tau \cap \sigma)) \right] \\ \forall p \in \mathcal{P}, 1 - 4G_N \int ds \, |U_\perp| - \int d\rho d\kappa N(\tau, p) \leq 0 \tag{3.4}$$

In the $U'$ constraint the $d\kappa(\sigma)$ can be integrated to 1 and similarly $d\rho(\tau)$ in the $V'$ constraint.

**Constraint interpretation**: In $U'$ threads/flows, when we take spacelike curves **without endpoints** we get usual norm bound as $N(\tau, p) = 0$ for such curves. The constraint **essentially forces** the $U'$ threads to go only in the future and past of the optimal surface of intersection of Cauchy slices and timesheets.

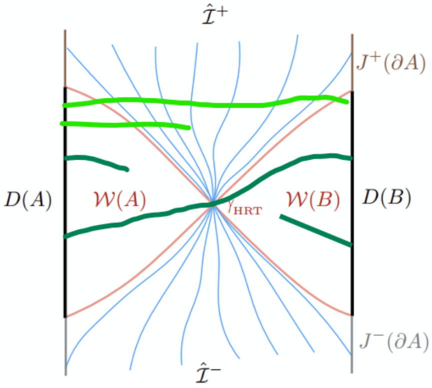



Figure: Think green spacelike curves saturate the constraint, but the light green spacelike curves don't.

# Flow Dualization

We start with the relaxed QES. For $\phi$, the level sets are Cauchy slices $\sigma_t$ and for $\psi$ the level sets are time sheets $\tau_s$.

$$f[\phi,\psi] := \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \left[ \frac{\text{Area}(\sigma_t \cap \tau_s)}{4G_N} + S_{\text{bulk}}(\sigma_t \cap \tau_s) \right]$$

$$f[\phi,\psi] = \frac{1}{4G_N} \int_{\mathcal{M}} \sqrt{g} |d\phi \dot{\wedge} d\psi| + \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \left[ S_{\text{bulk}}(\sigma_t \cap \tau_s) \right]$$

A generalization of the coarea formula was used for area. We know dualize from $\psi$ to $V$, with fixed $\phi$.

$$\text{minimize } \frac{1}{4G_N} \int_{\mathcal{M}} \sqrt{g} \mid d\phi \dot{\wedge} d\psi \mid + \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \left[ S_{\text{bulk}}(\sigma_t \cap \tau_s) \right]$$

$$\text{over } \psi \text{ such that } \quad \psi|_{D(A)} = -\frac{1}{2}, \quad \psi|_{D(B)} = \frac{1}{2}$$

Rewrite it by introducing a 1-form $X$

$$\text{minimize } \frac{1}{4G_N}\int_{\mathcal{M}}\sqrt{g}|d\phi\dot{\wedge}X| + \int_{-1/2}^{1/2}dt\int_{-1/2}^{1/2}ds\,[S_{\text{bulk}}(\sigma_t\cap\tau_s)]$$

$$\text{over } \psi, X \text{ such that } d\psi = X, \quad \psi|_{D(A)} = -\frac{1}{2}, \quad \psi|_{D(B)} = \frac{1}{2}.$$

Impose the constraint $d\psi = X$ using a Lagrange multiplier 1-form $V$.

$$L[\psi, X, V] = \frac{1}{4G_N}\int_{\mathcal{M}}\sqrt{g}[|d\phi\dot{\wedge}X| - 4G_N V\cdot(X - d\psi)] + \int_{-1/2}^{1/2}dt\int_{-1/2}^{1/2}ds\,[S_{\text{bulk}}(\sigma_t\cap\tau_s)]$$

$$L[\psi, X, V] = \int_{\mathcal{M}}\sqrt{g}[\frac{1}{4G_N}|d\phi\dot{\wedge}X| - V\cdot X] - \int_{\mathcal{M}}\psi d*V + \int_{-1/2}^{1/2}dt\int_{-1/2}^{1/2}ds\,[S_{\text{bulk}}(\sigma_t\cap\tau_s)]$$

$$-\int_{\mathcal{I}}\psi(*V) + \frac{1}{2}\left(\int_{D(A)}*V - \int_{D(B)}*V\right)$$

For the first term to be bounded we need $d\phi \pm 4G_N V \in \text{j}^+$. $*V|_{\mathcal{I}} = 0$ also follows from boundedness.

$$\Rightarrow \begin{aligned} L[\psi, X, V] = &- \int_{\mathcal{M}} \psi d * V + \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \left[ S_{\text{bulk}}(\sigma_t \cap \tau_s) \right] \\ &+ \frac{1}{2} \left( \int_{D(A)} *V - \int_{D(B)} *V \right) \end{aligned}$$

Define a simple function that is 1 on the side of $\tau_s$ that contains $D(A)$.

$$\chi_{\tau_s} = \begin{cases} 0 & \psi(x) > s \\ 1 & \psi(x) < s \end{cases}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \chi_{\tau_s} ds = \int_{\psi(x)}^{\frac{1}{2}} \chi_{\tau_s} ds = \int_{\psi(x)}^{\frac{1}{2}} 1 ds = \frac{1}{2} - \psi(x)$$

Replace $\psi(x)$ in the above Lagrangian with this.

$$L[\psi, X, V] = -\frac{1}{2}\int_{\mathcal{M}} d*V + \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \int_{\mathcal{M}} \chi_{\tau_s} d*V + \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds\, [S_{\text{bulk}}(\sigma_t \cap \tau_s)] + \frac{1}{2}\left(\int_{D(A)} *V - \int_{D(B)} *V\right)$$

Using $\int_{\mathcal{M}} d*V = -\int_{D(A)} *V - \int_{D(B)} *V$ it simplifies to

$$L[\psi, X, V] = \int_{D(A)} *V + \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \int_{\mathcal{M}} \chi_{\tau_s} d*V + \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds\, [S_{\text{bulk}}(\sigma_t \cap \tau_s)]$$

Now the above equation has 2 bulk terms. In the classical case the $S_{\text{bulk}}$ wont be there. We want to combine both of those terms and get the divergence constraint that we guessed as an **explicit constraint**.

$$\Rightarrow = \max_{\phi} \max_{V} \left[\int_{D(A)} *V + \min_{\psi} \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \left[\int_{D(A)\text{ side of }\tau_s} d*V + S_{\text{bulk}}(\sigma_t \cap \tau_s)\right]\right]$$

The optimal solution for the above program will be $(\psi^0, \phi^0, V^0)$. $\psi^0$ will be a delta that gives the same timesheet $\tau^0$ for all $s$. Similarly $\phi^0$. Let $\gamma^0$ be their intersection.

We want to show that all the optimal solutions for the above program are such that the optimal time sheet will be trivial which means inside it, it has the entanglement wedge and between the entanglement wedge and the optimal time sheet there is no flow.
Doing this is actually simple and follows from the norm bound. Even in the classical case we get the below 3 conditions.

$$\int_\tau *V \leq \int_{-1/2}^{1/2} dt \; area(\gamma_t)$$

$$4G_N \int_q *V \leq \phi(y) - \phi(x)$$

where $q$ is any timelike curve passing though $\gamma$.

$$V|_{I^\pm(\gamma^0)} = 0$$

The above equation tells that once the norm bound is saturated on the optimal $\gamma^0$ then $I^\pm(\gamma^0)$ **cannot have any non zero flow**. So any optimal timesheet is trivial and can be pushed towards the entanglement wedge.

# Entanglement distribution functions

An EDF is a function $f$ on $M$ such that, for all $A$ in the CFT,

$$\left| \int_{D(A)} f \right| \leq S(A)\,. \tag{3.5}$$

In the case of holography, the boundary entanglement distribution functions are **dual** to the bulk flows with $f = g(V, n) = V \cdot n$. Optimal flows give those boundary entanglement distribution functions that saturate for $D(A)$.
In the case of finite degrees of freedom, the function $f$ will be summed over the subsystems.

# Entropohedron

If the discrete degrees of freedom are $A_1, A_2, \cdots$ then $(f(A_1), f(A_2), \dots)$ is a vector that can be plotted. This Entropohedron encodes all the entanglement structures of the subsystems, just like the set of all $f$s.

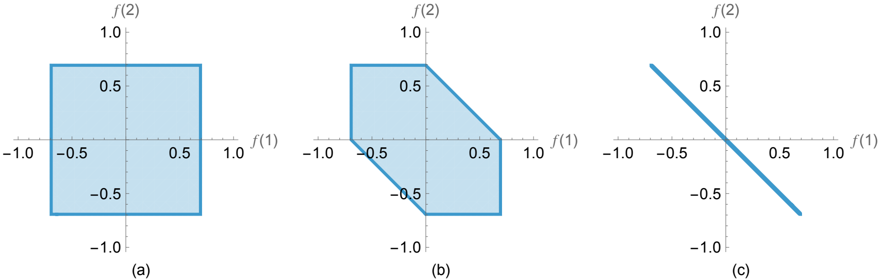



Figure: Entropohedra for various states on two qubits: (a) maximally mixed, $\rho = \frac{1}{4}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$. (b) Maximally classically correlated, $\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. (c) Maximally entangled (Bell pair), $\rho = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$.

# Entropohedron

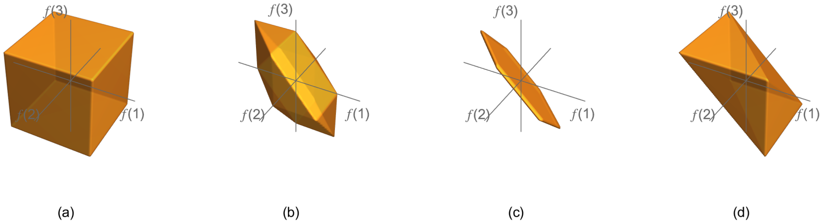



Figure: Entropohedra for various states on 3 qubits: (a) maximally mixed. (b) Maximally classically correlated. (c) Maximally entangled (e.g. GHZ or W state). (d) Marginal of four-party perfect tensor.

# Numerical

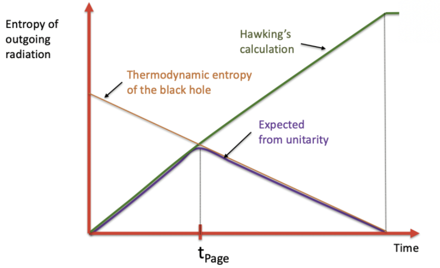
If we recall QES has a minimax and maximin formula in terms of $S_{gen}$ and **not a pure minimization or maximization**. If we want to do numerical calculations, convex/concave functions are much easier to deal with.
For $V$ and $U$ flows every time we guess a non optimal flows it gives an upper or lower bound. If we have more number of non optimal flows the bounds become stronger.

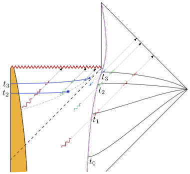
# Deriving Einstein's equations

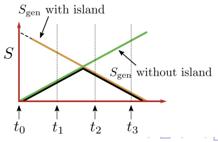
Agón Cáceres Pedraza 2020 derived (classical) Einstein's equations from bit threads following previous work where they derived from RT formula by Raamsdonk, Faulkner, Hartman etc. Their proof was a generalization of Ted Jacobson's old proof. In the RT formula version, the first law of entanglement entropy is mapped to Einstein's equations.
Earlier, we considered the metric as fixed, and then for all flows (even non-optimal), we demanded they satisfy the norm bound. We can **do the reverse**, for every optimal solution, the metric should satisfy the norm bound which implies the bulk gravity is classical GR. This was done for classical GR. If we do for QCBT, then we will **maybe** (we didn't try) derive semi-classical Einstein's equation with $\langle \hat{T}_{\mu\nu} \rangle$.

# Islands

Images from the review: Almheiri Hartman Maldacena Shaghoulian Tajdini 2020.

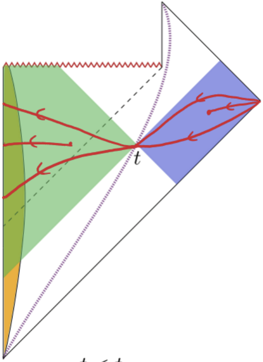


# Interior DoF are not independent

$$S(\mathrm{R}) = \underset{I}{\mathrm{ext}}\, S_{\mathrm{gen}}(\mathrm{R} \cup I) = \underset{I}{\mathrm{ext}} \left[ \frac{A(\partial I)}{4G} + S_{\mathrm{bulk}}(\mathrm{R} \cup I) \right]$$

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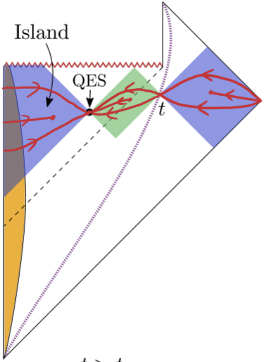
$(a)$

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$(b)$