Prepared for submission to Weekly Shōnen Jump

Fundamental Physics

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Abstract: These are some short notes on fundamental physics. I will try to discuss all topics related to classical mechanics, QM, GR, and QFT. These notes only talk about frameworks already experimentally verified. Things like inflation and beyond the Standard Model topics, etc, which are not yet experimentally verified but use these established frameworks are also included in this. These notes are useful only to ***revise*** stuff that you already know. For first-time learning, check the references cited under each section and assume that the first reference is the best.

For quantum gravity, check notes at ksr.onl/QG.

Last updated on June 9, 2025.

Currently, these notes are in the beginning stage & are mainly useful to see the references cited under each section.

Contents

1 Introduction

> The supreme task of the physicist is to arrive at those universal elementary laws from which the **cosmos can be built up by pure deduction**. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them. In this methodological uncertainty, one might suppose that there were any number of possible systems of theoretical physics all equally well justified; and this opinion is no doubt correct, theoretically. But the development of physics has shown that at any given moment, out of all conceivable constructions, a single one has always proved itself decidedly superior to all the rest.
>
> *Albert Einstein* (1918)

Fundamental physics is that part of physics that cannot be reduced to some other physics. If you keep asking "*Why?*" for physical phenomena, you always end up at fundamental physics, and your curiosity must end there as you can't find an answer to "*Why?*" anymore. By definition, the fundamental laws cannot be explained from some other physical explanation. The only type of rational[1] explanation I can think of is an ontological argument, you can argue that the fundamental laws of physics (probably a theory of everything) is the greatest possible (Platonic mathematical) entity, and due to this property it not existing physically is ***logically impossible*** and therefore it must physically exist *a priori* without any physical reason for its existence. Fundamental physics is essentially the Big Bad of physics, and it is the worst nightmare of an anarchist, an ultimately unquestionable authority that doesn't allow us to have freedom (free will).

Any reasonable scientist *must believe in the reductionist philosophy* that every physical phenomenon[2] can be reduced to fundamental physics. In philosophical words, all physical phenomena supervene on the fundamental laws of physics (probably a theory of everything). In [1], Anderson **correctly** points out that not everyone who agrees with reductionism must agree that fundamental physics is the most important research direction and explains that you can do highly creative research on emergent phenomena without working in fundamental physics. Anderson also mentions that many great fundamental physicists have used condescending language[3] to describe applied physics. I want to clarify that even though I am only interested in fundamental physics, I respect all fields and all researchers. My preference for fundamental physics is like my preference for our Indian cuisine compared to other cuisines or my preference for animanga (collective term for anime and manga) over other forms of entertainment, just a personal preference, and I don't claim things I like to be objectively more interesting.

What is considered fundamental physics depends on the time. For example, in Newton's time, his theories *Classical Mechanics* (CM) and *Newtonian Gravity* (NG) were con-

[1] This excludes religions as valid explanations because science assumes methodological naturalism, and we are compelled to believe in the full naturalism to explain the unreasonably consistent success of scientific predictions. Without naturalism, science is an unexplainable miracle.

[2] Including emergent phenomena we haven't yet understood, such as confinement or consciousness.

[3] "*the discoverer of the positron said "the rest is chemistry"*" here Anderson is talking about the *predictor* P. A. M. Dirac, not the *discoverer* Carl Anderson.

sidered fundamental. But today, we know that those 2 are some limits of quantum field theory ($QFT \to QM \to CM$) and general relativity ($GR \to NG$). QFT and GR are true, but in the future, we will find that they are only limiting cases of the theory of everything (the most promising candidate being string theory), and then that ***truest*** theory will replace these two as the fundamental theory. Lagrange, who rewrote Newton's theories into his new Lagrangian formulation, famously said,

> Newton was the greatest genius that ever existed, and the most fortunate, for we cannot find more than once a system of the world to establish.
>
> *Joseph-Louis Lagrange*

But Lagrange wrongly thought that fundamental physics was over with CM and NG. **Fortunately for us**, it's not over yet, and the last piece of fundamental physics is likely quantum gravity.

The theories considered in these notes are often stated to be "well-understood" when talking about quantum gravity. But that is obviously an overstatement. Even natural numbers (i.e., Peano axioms) have many unsolved[4] problems like odd perfect number (open from AD 100), Goldbach's conjecture (open from 1742), etc. Even the simplest physical theories are much more complicated than natural numbers. The three-body problem in Newtonian gravity has no general closed-form solution, and the only way to solve it is to calculate using **approximate** numerical methods. We don't know if Feigenbaum constants are transcendental or even irrational. Turbulence is also not understood. This is all in Newtonian physics. In general relativity, even the two-body problem itself is unsolvable. And for QFT, it's much worse; we don't even know how to rigorously define them; check section 7.26.

In practice, we must study the limiting cases before studying the more fundamental QFT and GR. [59–71] are books that are similar to these notes. [59–70, 74] follows the same approach as me, the bottom-up approach, starting with old fundamental theories and ending with QFT. [71] follows the reverse approach starting with $QFT \to QM \to CM$. I am not saying these are the best books to study QFT, which is definitely not true. I am merely pointing out that these books are organized similarly to these notes[5]. Also, check [72, 73] for short notes related to fundamental physics and [6, 54] for relevant mathematics notes. Also, [74, 75] are good reviews of undergraduate physics.

[4]Unfortunately, there are also limitations to mathematics because of the 2 Gödel's incompleteness theorems, as mentioned at the start of section 2. Check statements independent of ZFC [82, 83] and undecidability in number theory [84]. These problems are worse than unsolved, and you can equally believe they are either true or false and add it as an axiom. See [85–87] for undecidable problems in QFT.

[5]Feynman said if "all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures", he would pass "all things are made of atoms". If it was an entire book, it should be Penrose's [63].

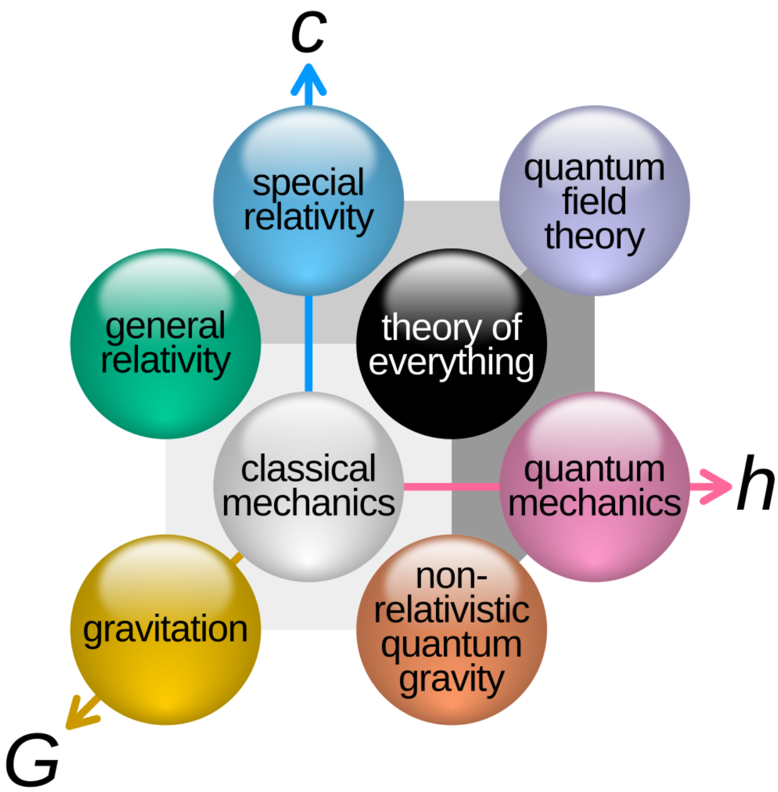


Figure 1. Different limits of fundamental physics. Section 3 starts discussing the origin and the G-axis is in sections 3.1.1 & 4.2.1 and the $\frac{1}{c}$-axis (image incorrectly calls it c-axis) is in section 3.8. $G\frac{1}{c}$-plane is in section 5. The h-axis is in section 6. The ch-plane is in 7. We do not discuss the non-relativistic quantum gravity limit (Gh-plane).
Note that there are several things missing in the diagram. For example, if we take the standard model on the QFT plane then at the origin ($h \to 0$ and $\frac{1}{c} \to 0$ limit) apart from classical mechanics we will also have Galilean Electromagnetism 4.4.6. The $\frac{1}{c}$-axis also contains the classical electromagnetic field 4.4.

History: Originally, to satisfy their curiosities, humans made up deities to explain all physical phenomena, such as deities for water, air, thunder, and fire, etc. or in some places, even a single deity to explain all physical phenomena; see this funny video. It would be vacuously wrong to keep saying 'God Did It' for every physical phenomenon. Thales (Milesian school) was the first[6] recorded person to argue that we can explain physical phenomena using natural laws without invoking the whims of deities. Aristotelian physics was an attempt to find qualitative natural laws of physics that do not need deities. From Galileo's time, people were no longer satisfied with the qualitative non-mathematical laws of physics. Once physics started using/developing mathematics, progress happened at a higher rate, and classical mechanics, thermodynamics, electromagnetism, etc, were discovered in the next few centuries. Then, in the 1900s, progress in physics happened even

[6] The Cārvāka also believed this around the same time & they even went one more step and proclaimed that the approach to find these natural laws is empiricism.

faster, and general relativity, quantum mechanics, QFT, etc, were discovered. The current status is shown in figure 1, the planes $h\frac{1}{c}$ and $G\frac{1}{c}$ are well understood. I will not discuss more history of physics; you can check [78–81] for that.

Approach: The main approach is empiricism: as Feynman famously compared, if we think of nature/universe as a big chess board, then by looking at the moves (empirical data) for a long enough time, we can guess the rules that govern these moves & the mathematical language needed to write them. But there have been theoretical advances in physics that happened **without input from empirical data**, with the chief examples being Riemannian geometry and Einstein's General Relativity.

> A great deal more was hidden in the Dirac equation than the author had expected when he wrote it down in 1928. Dirac himself remarked in one of his talks that his equation was more intelligent than its author.
>
> *Victor Weisskopf*

Remark 1 (The far-reaching power of equations)**.** In physics, the equations often tell much more information about reality than what the discoverer anticipated. The conservation of energy in Newtonian mechanics was not understood till Émilie du Châtelet. Newton didn't know that Newtonian spacetime is a fiber bundle [174]. Both vectors and differential forms were discovered after the death of Maxwell, meaning his understanding of Maxwell's equations was limited, and he didn't know things like Maxwell's equations being Lorentz invariant. Woldemar Voigt, George FitzGerald, Joseph Larmor, and Hendrik Lorentz all proposed the same Lorentz transformation equations, but they all interpreted them wrong until Einstein gave the correct Special Relativity interpretation, and even Einstein not only didn't realise the full geometric interpretation given by Minkowski he even said "Since the mathematicians have invaded the theory of relativity, I do not understand it myself any more" but later in his decade long quest for General Relativity he understood the importance of the geometric picture. Dirac didn't interpret his equations as a quantum field theory when he first proposed it. This is why hero worship is bad for physics compared to equation worship.

Remark 2 (Theory)**.** In most scientific fields, the word 'theory' is usually reserved for only those hypotheses that are empirically verified. But in fundamental physics, this word is replaced with 'phenomenological theory'. ϕ^4 field theory, Yang-Mills theory, etc, are called 'formal theories'.

> If nature were not beautiful it would not be worth knowing, and life would not be worth living.
>
> *Henri Poincaré*

2 Mathematics

> Philosophy is written in this grand book, which stands continually open before our eyes (I say the 'Universe'), but can not be understood without first learning to comprehend the language and know the characters as it is written. It is written in mathematical language, and its characters are triangles, circles and other geometric figures, without which it is impossible to humanly understand a word; without these one is wandering in a dark labyrinth.
>
> *Galileo Galilei* (1623)

[4–54]. Apart from these books, also see these playlists [55–57]. Main reference is [4]. See these for very compact and concise reviews of the maths needed for physics: [51, 52], chapter 14 of [369] and chapters 1 to 7 of [478].

There is no rigorous definition of what maths is. On page 2 of [88], Thurston defined it recursively using mathematicians. I will give the below decent definition.

Definition 1 (Mathematics)**.** Mathematics is *a priori* knowledge that is rigorous or precise.

By this definition, Newton's calculus[7] was proto-maths until real analysis came. From '*a priori*', it follows that mathematical objects are abstract (i.e., do not exist in spacetime). Any statement of the form "If actually p, then p" is an *a priori* logically true statement, but if p cannot be rigorously and abstractly defined, then it is not maths. As truth requires precision, mathematics is the most truthful subject. For example, "*If abusing a sentient being lifelong for sensual palate pleasure of one meal is immoral, then carnism is immoral*" is *a priori* true without knowing any empirical data, but it is not a mathematical statement as we cannot (yet) rigorously define abstract concepts like 'moral'.

> Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.
>
> *Bertrand Russell* (1901)

> If a 'religion' is defined to be a system of ideas that contains unprovable statements, then Gödel taught us that mathematics is not only a religion, it is the only religion that can prove itself to be one.
>
> *John D. Barrow*

Long ago, mathematical theories were heuristic. At the end of the 19th century, due to inconsistencies like Russell's paradox, there was a crisis in the foundations of mathematics. Hilbert's program was a proposed solution for the crisis. Often, we can relate mathematical theories as stronger/weaker; for example, ZFC set theory is stronger than

[7] Although Newton was the dominant contributor, there were many others who contributed to calculus, such as Mādhava, who found Taylor series centuries before Taylor.

Peano Arithmetic [89]. The existence of a model of Peano Arithmetic within ZFC and its consistency can be proved within ZFC [90]. We don't even need the stronger ZFC; for example, Gentzen's consistency proof uses Primitive Recursive Arithmetic, which is neither weaker nor stronger than PA. Hilbert's program was to prove the consistency of stronger theories using weaker theories. This way, Hilbert wanted to reduce all mathematics to a finite, complete, and consistent set of axioms. Gödel's 2nd incompleteness theorem proved that Hilbert's goal is ***impossible***. It says that we can't prove the consistency of a mathematical theory *within that theory itself* if the theory contains basic arithmetic, so if such a theory can't even prove its own consistency, then obviously, it can't prove a stronger theory's consistency. So, the kind of reductionism that Hilbert envisioned is unfortunately impossible. In the aftermath of Gödel, Hilbert's program was divided into many less ambitious directions like *proof theory*, *reverse mathematics* etc.

We have many proofs that Peano Arithmetic is consistent even though none of them uses a weaker theory [91]. This gives a lot of theoretical evidence that Peano Arithmetic is consistent because if it is not consistent, then all of those different theories within which it was proved must also be inconsistent. So, the amount of faith to believe in its consistency is very tiny compared to, for example, faith in religion or even the scientific method. Interestingly, Peano arithmetic allows non-standard model of arithmetic, but Peano axioms[8] only allow natural numbers.

All mathematical truths can be considered tautologies. "$1+1=2$" might not look like a tautology. But we can always append "If we take the relevant definitions and axioms and assume their consistency, then" at the beginning of a mathematical statement to make it a tautology. The relevant definitions and axioms can be Peano axioms or ZFC set theory etc.

We might come back to the foundations in 2.10, but we will take the foundations for granted for now. From this point on, we shall assume the consistency of ZFC set theory, and it will be more than enough for physics. Even after believing in its consistency, there will be many undecidable problems [82–86].

1. Foundations (Logic, Set theory, Category theory, Univalent foundations, etc)
2. Analysis (Real analysis, Complex analysis, Functional analysis/Calculus of variations, Measure theory, etc)
3. Algebra (Groups, Rings, Fields, Homological algebra, etc)
4. Geometry and topology (Riemannian geometry, Discrete geometry, Symplectic geometry, General topology, Differential topology, Homotopy theory, etc)
5. Number theory (Arithmetic, Prime numbers, Diophantine equations, etc)
6. Applied math (Computational mathematics, Statistics, Mathematical physics, etc)

[8] See this nice game.

Mathematics can be broadly divided into the above 6 subfields. There are many subfields that mix these subfields, such as algebraic topology, algebraic geometry, algebraic number theory, etc. The main areas that are needed for physics are 2, 3, and 4.

> Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.'
>
> *Michael Atiyah*

Geometry & Topology: In both Geometry & Topology, we study mathematical spaces that have various amounts of structure defined on them; see Figs 2 & 3. Geometry studies mathematical spaces' local (or infinitesimal) structure, such as metric properties. Topology studies the global structure of mathematical spaces.

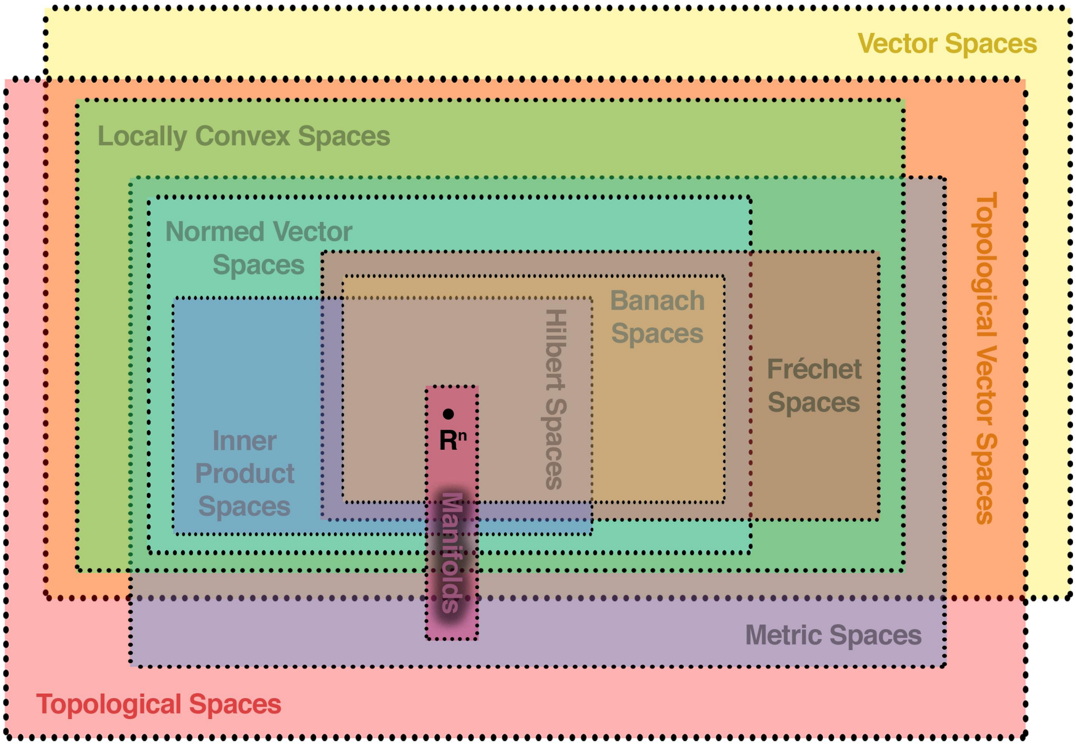


Figure 2. Some popular mathematical spaces. $\mathbb{R}^n$ is an example of all of these spaces. There is no mathematical reason why there is a black color behind the word Manifolds; it's only to make that word readable. Source.

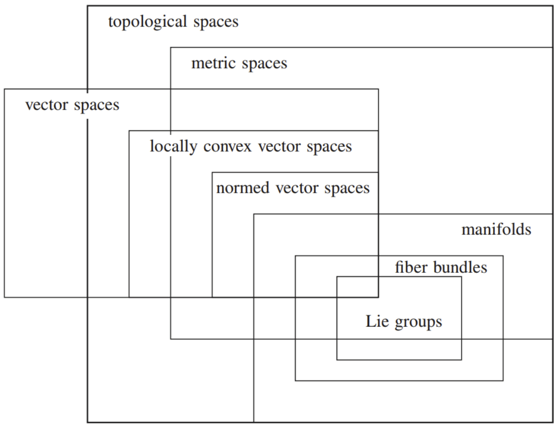


Figure 3. Another similar image of mathematical spaces that includes Lie groups & fiber bundles from [32]. Orbifolds, orientifolds, Lie groupoids, etc, can also be added.

Abstract algebra: Abstract algebra studies algebraic structures, which are sets with specific operations acting on their elements.

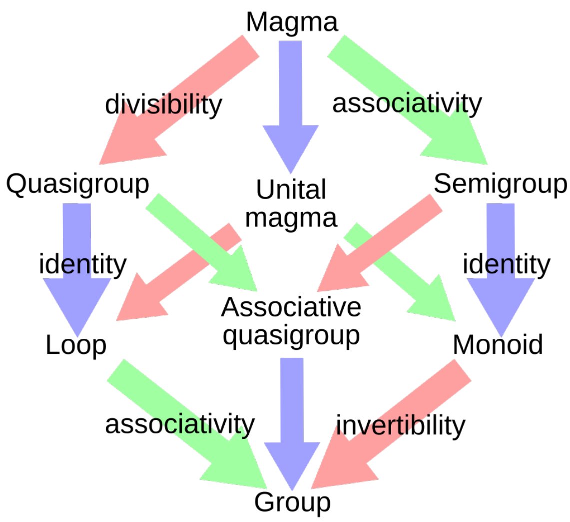


Figure 4. Some algebraic structures

It will be an overkill to start from scratch like in the legendary *Principia Mathematica* where they took 379 pages to prove $1 + 1 = 2$. For the sake of physics, we can believe in the mathematicians and take the foundations for granted. I will assume JEE level maths as preliminaries and 2.1 has some important formulas on these topics.

2.1 Mathematical preliminaries

2.1.1 Matrices

$$\det(\mathbf{A}) = \varepsilon_{i_1 \ldots i_n} a_{1i_1} \ldots a_{ni_n},$$

$$\det(\mathbf{A}) = \frac{1}{n!}\varepsilon_{i_1\ldots i_n}\varepsilon_{j_1\ldots j_n}a_{i_1j_1}\ldots a_{i_nj_n},$$

$$\sum_{i_1,i_2,\ldots}\varepsilon_{i_1\ldots i_n}a_{i_1\,j_1}\ldots a_{i_n\,j_n} = \det(\mathbf{A})\varepsilon_{j_1\ldots j_n}$$

$$d\det(A) = \operatorname{tr}(\operatorname{adj}(A)\,dA) = \det(A)\operatorname{tr}\left(A^{-1}dA\right)$$

$$\det(e^A) = e^{\operatorname{tr}(A)}$$

$$\det(A) = e^{\operatorname{tr}(\ln A)}$$

$$\begin{aligned}
&\det(I+tA)\\
=&\exp\left(\operatorname{Tr}\log(I+tA)\right)\\
=&\exp\left(t\operatorname{Tr}A - \frac{t^2}{2}\operatorname{Tr}A^2 + \frac{t^3}{3}\operatorname{Tr}A^3 + \cdots\right)\\
=&1 + t\operatorname{Tr}A + \frac{t^2}{2!}\left((\operatorname{Tr}A)^2 - \operatorname{Tr}A^2\right) + \frac{t^3}{3!}\left((\operatorname{Tr}A)^3 - 3(\operatorname{Tr}A)(\operatorname{Tr}A^2) + 2\operatorname{Tr}A^3\right) + \cdots + t^n\det(A)
\end{aligned}$$

where A is a $n \times n$ matrix.

2.1.2 Integration

Gaussian integrals

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\int_0^{\infty} e^{-r^2}\,2\pi r dr} = \sqrt{\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-a(x+b)^2}\,dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n}A_{ij}x_ix_j\right)d^nx = \int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}x^{\mathsf{T}}Ax\right)d^nx = \sqrt{\frac{(2\pi)^n}{\det A}} = \sqrt{\det(2\pi A^{-1})}$$

Gamma function

Beta function

Below is a nice nonrigorous trick

$$\int e^{ax}e^{ibx}dx = \frac{e^{(a+ib)x}}{a+ib} + C = \frac{e^{ax}e^{ibx}(a-ib)}{a^2+b^2} + C$$

$$\Rightarrow \int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$$

$$\Rightarrow \int e^{ax}\cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C$$

2.1.3 Vector differentiation

Some vector calculus identities.

Gradient

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla(\psi\mathbf{A}) = \nabla\psi \otimes \mathbf{A} + \psi\nabla\mathbf{A}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

Divergence

$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{A}$$

Curl

$$\nabla\times(\psi\mathbf{A}) = \psi\,(\nabla\times\mathbf{A}) - (\mathbf{A}\times\nabla)\psi = \psi\,(\nabla\times\mathbf{A}) + (\nabla\psi)\times\mathbf{A}$$

$$\nabla\times(\psi\nabla\phi) = \nabla\psi\times\nabla\phi$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\,(\nabla\cdot\mathbf{B}) - \mathbf{B}\,(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\,\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B}$$

Vector dot Del Operator

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = \frac{1}{2}\Big[\nabla(\mathbf{A}\cdot\mathbf{B}) - \nabla\times(\mathbf{A}\times\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla\times\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + \mathbf{A}(\nabla\cdot\mathbf{B})\Big]$$

Second derivatives

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\psi) = \mathbf{0}$$

$$\nabla\cdot(\nabla\psi) = \nabla^2\psi\text{(scalar Laplacian)}$$

$$\nabla\,(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) = \nabla^2\mathbf{A}\text{(vector Laplacian)}$$

$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

$$\psi\nabla^2\phi - \phi\nabla^2\psi = \nabla\cdot(\psi\nabla\phi - \phi\nabla\psi)$$

$$\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2(\nabla\phi)\cdot(\nabla\psi) + \left(\nabla^2\phi\right)\psi$$

$$\nabla^2(\psi\mathbf{A}) = \mathbf{A}\nabla^2\psi + 2(\nabla\psi\cdot\nabla)\mathbf{A} + \psi\nabla^2\mathbf{A}$$

$$\nabla^2(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\nabla^2\mathbf{B} - \mathbf{B}\cdot\nabla^2\mathbf{A} + 2\nabla\cdot((\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A}))\text{(Green's vector identity)}$$

Third derivatives

$$\nabla^2(\nabla\psi) = \nabla(\nabla\cdot(\nabla\psi)) = \nabla\left(\nabla^2\psi\right)$$

$$\nabla^2(\nabla\cdot\mathbf{A}) = \nabla\cdot(\nabla(\nabla\cdot\mathbf{A})) = \nabla\cdot\left(\nabla^2\mathbf{A}\right)$$

$$\nabla^2(\nabla\times\mathbf{A}) = -\nabla\times(\nabla\times(\nabla\times\mathbf{A})) = \nabla\times\left(\nabla^2\mathbf{A}\right)$$

2.1.4 Vector integration

0d and 1d

$$\psi|_{\partial P} = \psi(\mathbf{q}) - \psi(\mathbf{p}) = \int_P \nabla\psi \cdot d\boldsymbol{\ell}(\text{gradient theorem})$$

$$\mathbf{A}|_{\partial P} = \mathbf{A}(\mathbf{q}) - \mathbf{A}(\mathbf{p}) = \int_P (d\boldsymbol{\ell} \cdot \nabla)\,\mathbf{A}$$

1d and 2d

$$\oint_{\partial S} \mathbf{A} \cdot d\boldsymbol{\ell} \;=\; \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}(\text{Stokes' theorem})$$

$$\oint_{\partial S} \psi\, d\boldsymbol{\ell} \;=\; -\iint_S \nabla\psi \times d\mathbf{S}$$

$$\oint_{\partial S} \mathbf{A} \times d\boldsymbol{\ell} \;=\; -\iint_S (\nabla\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{1}) \cdot d\mathbf{S} \;=\; -\iint_S (d\mathbf{S} \times \nabla) \times \mathbf{A}$$

2d and 3d

$$\psi\, d\mathbf{S} \;=\; \iiint_V \nabla\psi\, dV$$

$$\mathbf{A} \cdot d\mathbf{S} \;=\; \iiint_V \nabla \cdot \mathbf{A}\, dV(\text{divergence theorem})$$

$$\mathbf{A} \times d\mathbf{S} \;=\; -\iiint_V \nabla \times \mathbf{A}\, dV$$

$$\psi\nabla\varphi \cdot d\mathbf{S} \;=\; \iiint_V \left(\psi\nabla^2\varphi + \nabla\varphi \cdot \nabla\psi\right)\, dV(\text{Green's first identity})$$

$$=\; \iiint_V \left(\psi\nabla^2\varphi - \varphi\nabla^2\psi\right)\, dV(\text{Green's second identity})$$

$$\psi\mathbf{A} \cdot d\mathbf{S} - \iiint_V \psi\nabla \cdot \mathbf{A}\, dV(\text{integration by parts})$$

$$\psi\mathbf{A} \cdot d\mathbf{S} - \iiint_V \mathbf{A} \cdot \nabla\psi\, dV(\text{integration by parts})$$

$$(\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{S} + \iiint_V (\nabla \times \mathbf{A}) \cdot \mathbf{B}\, dV(\text{integration by parts})$$

2.1.5 Coordinate geometry

> I have come to think that these writers themselves, with a kind of pernicious cunning, later suppressed this mathematics as, notoriously, many inventors are known to have done where their own discoveries were concerned. They may have feared that their method, just because it was so easy and simple, would be depreciated if it were divulged; so to gain our admiration, they may have shown us, as the fruits of their method, some barren truths proved by clever arguments, instead of teaching us the method itself, which might have dispelled our admiration.
>
> *René Descartes* (Source)

Coordinate geometry simplified the complicated coordinate-free geometrical proofs of ancient mathematicians (like Apollonius of Perga's conic sections proofs) to such a great extent that René Descartes believed in a conspiracy that ancient mathematicians hid this approach to make mathematics look harder/prestigious.

Coordinate geometry is arguably the first idea of ***fake degrees of freedom***, which is routinely used in gauge theory. To describe a polygon, we need $2n-3$ parameters, out of which the number of sides given should be n or $n-1$ or $n-2$, and the remaining should be angles. This is related to the fact all triangulations of an n-gon will have exactly $n-2$ triangles. When we use coordinate geometry on the plane $\mathbb{R}^2$, the number of parameters becomes $2n$. So, we have introduced 3 fake degrees of freedom into this geometrical problem of studying a polygon. These 3 degrees of freedom correspond to the 2 position degrees of freedom of the origin and 1 angle degrees of freedom of the rotations of axes. We can either "interpret" these fake degrees of freedom actively or passively. At the end of the day, the only real invariant quantities, in this case, are lengths and angles. This is a very simple case where we are only considering global fake degrees of freedom. We could also introduce local/gauge fake degrees of freedom by considering more general coordinates/metrics (like polar coordinates) rather than only Cartesian coordinates.

2.1.6 Maps

Definition 2 (Homomorphism)**.** It is a map that preserves the algebraic structure.

Example 1 (Projecting vectors)**.** Consider a vector space. Projecting vectors to a single axis will preserve the vector addition.

Definition 3 (Isomorphism)**.** It is a homomorphism that is bijective.

Example 2 (Complex numbers as matrices)**.** $z = a + bi$ can be expressed as $z = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. This is an isomorphism.

2.2 Analysis

2.2.1 Real analysis

The original calculus is not satisfactory because there are many intuitive things about infinitesimals that we subconsciously assume. But our intuition is not trustworthy. Examples (3), (4) & (5) show that human intuition is not trustworthy.

Example 3 (Non-analytic smooth function)**.** This function has continuous derivatives of all orders at the origin, yet it is not analytic at the origin.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \tag{2.2.1}$$

Example 4 (Borwein integral). This example shows that just because a pattern is repeating for many numbers doesn't mean it will keep holding.

$$\int_0^\infty 2\cos(x)\frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/111)}{x/111}\,dx = \frac{\pi}{2}, \tag{2.2.2}$$

$$\int_0^\infty 2\cos(x)\frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/111)}{x/111}\frac{\sin(x/113)}{x/113}\,dx \approx \frac{\pi}{2} - 2.3324\times 10^{-138}. \tag{2.2.3}$$

This integral is $\frac{\pi}{2}$ as long as $\frac{1}{3}+\frac{1}{5}+\cdots<2$. From 113, it starts changing. See this video by 3Blue1Brown.

Example 5 (Weierstrass function). This function is continuous everywhere but differentiable nowhere. If we integrate this, we get functions that are nth differentiable everywhere but nowhere $n+1$th differentiable.

$$f(x) = \sum_{n=0}^\infty a^n\cos(b^n\pi x),$$

where $0<a<1$, b is a positive odd integer, and

$$ab > 1+\frac{3}{2}\pi.$$

Definition 4 (Weierstrass' (ϵ,δ)-definition).

Theorem 1 (Fundamental theorem of calculus).

Example 6 (Dirichlet function).

$$\mathbf{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q}\end{cases}$$

Example 7 (Thomae's function).

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x=\frac{p}{q} \quad (x \text{ is rational}), \text{ with } p\in\mathbb{Z} \text{ and } q\in\mathbb{N} \text{ coprime} \\ 0 & \text{if } x \text{ is irrational.}\end{cases}$$

2.2.1.1 $\mathbb{R}^n$

Remark 3 (Higher dimensional differentiation facts). Some unintuitive facts.

i Differentiable at (x_0, y_0).

ii $\nabla_{\mathbf{v}} f(\mathbf{x}) = (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot \mathbf{v}$ is true for all unit vectors $\mathbf{v}$.

iii The directional derivative exists for all $\mathbf{v}$.

iv f_x and f_y exist.

v f is continuous at (x_0, y_0).

$$(i) \Rightarrow all$$

$$(v) \not\Rightarrow (iv) \not\Rightarrow (iii) \not\Rightarrow (ii) \not\Rightarrow (i)$$

Example (8) proves that $(iii) \not\Rightarrow (i)$ and $(iii) \not\Rightarrow (ii)$.

Example (9) proves that $(iii) \not\Rightarrow (ii)$, $(iii) \not\Rightarrow (i)$ and $(iv) \not\Rightarrow (v)$ and even $(iii) \not\Rightarrow (v)$.

Example (10) shows that $(ii) \not\Rightarrow (i)$.

Example 8 (Higher dimensional example 1). The following function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x,y) = \begin{cases} 0, & \text{if } y = 0, \\ \dfrac{y}{|y|}\sqrt{x^2+y^2}, & \text{if } y \neq 0. \end{cases} \tag{2.2.4}$$

is continuous but not differentiable at $(0,0)$. But all directional derivatives exist. $\nabla_{\mathbf{v}} f(\mathbf{x})$ is $\frac{v_y}{|v_y|}$ for $v_y \neq 0$ and is 0 for $\mathbf{v} = (\pm 1, 0)$. As always $\mathbf{v} = (v_x, v_y)$ is a unit vector.

Example 9 (Higher dimensional example 2). The following function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \tag{2.2.5}$$

is not continuous at $(0,0)$ (approach the origin along $y = x$ and $y = x^2$ this gives two different values, so discontinuous). Since $(i) \Rightarrow (v)$ it is also not differentiable. But all partial derivatives exist and are given by

$$\nabla_{\mathbf{v}} f(0,0) = \begin{cases} \frac{v_y}{|v_y|}, & v_y \neq 0 \\ 0, & v_y = 0 \end{cases}$$

Example 10 (Higher dimensional example 3). Define a function as $f : \mathbb{R}^2 \to \mathbb{R}$ and

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \tag{2.2.6}$$

Here, except (i), all others are true.

Remark 4 (Fractional calculus).

2.2.1.2 Nonstandard analysis

Surreal number
Hyperreal number

2.2.2 Complex analysis

Definition 5 (Wirtinger derivatives).

Remark 5. In what sense are z and $\bar{z}$ independent? In the below, even though $z, \bar{z}$ look independent just like a, b, notice that a, b must be real. On the complex plane, we have a preferred origin and a preferred basis of $1, i$. z and $\bar{z}$ are not independent as they are related by a mirror image, but it just so happens that while differentiating, we can treat them as independent.

$$\begin{cases} z = a + bi \\ z^* = a - bi \end{cases} \iff \begin{cases} a = \frac{1}{2}z + \frac{1}{2}z^* \\ b = \frac{1}{2i}z - \frac{1}{2i}z^* \end{cases} \tag{2.2.7}$$

2.2.2.1 Cauchy's integral formula

Theorem 2 (Cauchy's integral theorem).

Theorem 3 (Liouville's theorem).

2.2.3 Hypercomplex analysis

Definition 6 (Cayley–Dickson construction).

Remark 6 (Geometric algebra). [92] Geometric algebra is not a mathematical field. It is, at best, hype; at worse, a crackpot cult. They use well-known mathematics like Clifford algebra to simplify some things but complicate other things in mechanics and electromagnetism. People like David Hestenes think all high school mechanics should be taught in this. Here, we briefly review their tricks.

Rotor

2.2.3.1 Quaternions

2.2.3.2 Octonions

2.2.3.3 Grassmann numbers

$$f(\eta_1,\eta_2) = f(0,0) + \left.\frac{\partial f}{\partial \eta_1}\right|_0 \eta_1 + \left.\frac{\partial f}{\partial \eta_2}\right|_0 \eta_2 + \frac{1}{2}\left.\frac{\partial^2 f}{\partial \eta_1 \partial \eta_2}\right|_0 \eta_2\eta_1 + \frac{1}{2}\left.\frac{\partial^2 f}{\partial \eta_2 \partial \eta_1}\right|_0 \eta_1\eta_2 \quad (2.2.8)$$

$$= f(0,0) + \left.\frac{\partial f}{\partial \eta_1}\right|_0 \eta_1 + \left.\frac{\partial f}{\partial \eta_2}\right|_0 \eta_2 + \left.\frac{\partial^2 f}{\partial \eta_1 \partial \eta_2}\right|_0 \eta_2\eta_1 \quad (2.2.9)$$

2.2.4 Calculus of variations

Chapter 9 of [41] and chapter 11, 12 of [390] and chapter 17 of [38] and chapter 8 of [42] and chapter 5 of [49]

Definition 7 (Gateaux derivative).

Fréchet derivative

2.2.5 Measure theory

[93, 94]

2.2.5.1 Lebesgue measure

Example 11 (Vitali set).

2.2.5.2 Lebesgue integration

2.2.5.3 Probability theory

2.2.5.4 Random matrix theory

[95–97]

2.2.6 Functional analysis

[98–101] & part 1 of [342] & chapter 13 of [38] and brief reviews: Chapter 7 of [104] & [102, 103]

2.2.6.1 Spectral theory

2.2.6.2 Operator algebras

2.2.6.3 Functional/path integrals

[104–106]

2.2.7 Harmonic analysis

2.3 Algebra

2.3.1 Group theory

[52, 107–111]

Theorem 4 (Classification of finite-dimensional simple Lie algebras).

Theorem 5 (Classification of finite simple groups). [a]

[a] Its also called the enormous theorem, as it took 100s of authors & tens of 1000s of pages.

2.3.2 Representation theory

[52, 107–111]

2.3.2.1 $SO(n)$

Remark 7 (Intrinsic generators of a sphere). The 3 generators of $SO(3)$ symmetry of a 2-sphere are normally called rotations, but this answer is only valid once we embed it in a higher-dimensional flat space. The intrinsic interpretation is just the 2 translation symmetries and 1 rotation symmetry on the 2-sphere. For example, imagine you are an insect living at (0,0,1). If we rotate along the z-axis in the flat space in which the 2-sphere is embedded, that looks like a rotation to you, but if we rotate in any perpendicular axis like x or y, that will be a translation along some geodesic to you. Similarly, for an n-sphere, the $SO(n+1)$ symmetry contains n translations and the $SO(n)$ rotation group. So, on curved spaces, the interpretation of rotation vs translation depends on the observer.

2.3.2.2 $SO(1, d)$

[112–114]

Definition 8 ($SO(1, d)$).

See this nice page about the generators of boosts [115].

Definition 9 (Wigner's classification).

2.3.2.3 $SU(N)$

2.3.3 Ring theory

2.3.4 Linear algebra

2.4 Topology

[58, 116]

A topological space is a space that has a notion of 'open' and 'closed' sets.

Definition 10 (Topological spaces).

Definition 11 (Closed set).

Definition 12 (Clopen set).

Definition 13 (Compactness).

Definition 14 (Path connectedness).

Theorem 6 (Brouwer fixed-point theorem).

2.5 Category theory

[117, 118]

2.6 Algebraic topology

There is no book like [58], but see this stack exchange thread for counterexamples [119].

2.6.1 Homology groups

2.6.2 Homotopy groups

Definition 15 (Seifert-Van Kampen theorem).

2.6.3 de Rham cohomology groups

2.7 Differential geometry

2.7.1 Manifolds

Remark 8 (Exotic $\mathbb{R}^4$).

2.7.2 Riemannian geometry

2.7.3 Complex manifolds

2.7.4 Poisson geometry

[128, 129]

2.7.4.1 Symplectic geometry

[130]

2.7.5 Projective differential geometry

[5]

2.7.6 G and (G, X) and Cartan's (G, P)-structures

[131–134][5, 46][273]

2.8 Algebraic geometry

[135–137][17, 18]

Definition 16 (Torsors). [138]

2.8.1 Sheaf theory

[139]

2.9 Topology ∩ Geometry ∩ Algebra

2.9.1 Fibre bundles

2.9.2 Connections on fiber bundles

2.9.3 Characteristic classes

2.9.4 Spin structures

[140] and [44]

> Uh what is a spinor? Now I spent most of my life working on spinors in one form or another and I don't know. Only god knows maybe Dirac but he's no longer with us.
>
> *Michael Atiyah* (Source)

2.9.5 Index theorems

2.9.6 Differential cohomology

[145–147]

2.9.7 Condensed mathematics

[148, 149]

2.10 Foundations

[2, 150, 151]

2.10.1 Logic

2.10.2 ZFC, NBG, MK and TG set theories

[152, 153][2]

Theorem 7 (Well-ordering theorem).

Remark 9 (Well-ordering of $\mathbb{R}$ within ZFC). [154, 155]

2.10.3 ETCS

[156]

2.10.4 Proof theory

[157]

2.10.5 Reverse mathematics

[158–160]

2.10.6 Univalent foundations

Chapter 2 of [161].

Part I

Classical physics

3 Classical mechanics

[162–173]

In classical mechanics, we study point particles. These particles interact with classical fields like Newtonian gravity, electromagnetism, etc. Macroscopically, these interactions are complicated due to the large number of particles, but we can consider some approximations like *rigid bodies* that will simplify everything. For such rigid bodies, the complicated electromagnetic interactions can be idealized into the simple cases of *normal forces, friction*, etc.

> In order to put his system into mathematical form at all, Newton had to devise the concept of differential quotients and propound the laws of motion in the form of total differential equations—perhaps the **greatest advance in thought that a single individual was ever privileged to make**.
>
> *Albert Einstein*

3.1 Newtonian formulation

Definition 17 (Newtonian mechanics)**.** Forces acting on a particle are assumed to be vectors that act instantaneously. The net vector addition of all the forces acting on a particle is assumed to be proportional to the acceleration. We shall name the proportionality constant as mass.

$$\mathbf{F}(\mathbf{x}(t)) = m\frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2}$$

Any reference frame where the above vector equation works is defined as an **inertial reference frame**. We assume such reference frames exist. For noninertial reference frames, we can either transform back to some inertial reference frame and apply the above equation, or we can equivalently add pseudo forces.

Newtonian spacetime: [174, 175]

Remark 10. **Relation to more fundamental theories**: This is a brief remark on the worldline formalism, which is properly discussed in my QG notes. Newtonian mechanics is a map $x^i : \mathbb{R} \to \mathbb{R}^n$, where x^i are Klein-Gordon fields as explained in my QG notes. Here, $\mathbb{R}^n$ is called the **Euclidean target space**, and $\mathbb{R}$ is called **time**. Special relativity (Lorentzian target space), the Heisenberg picture of Quantum Mechanics, the worldline formalism of QFT & GR, and the worldsheet formalism of string theory are the natural generalizations of Newtonian mechanics. Just like all these theories have a formulation in the target space (Schrödinger picture, QFT, String Field Theory), Newtonian mechanics admits a similar formulation called the Koopman–von Neumann formulation 3.5. In fact, there is even an interaction picture for classical physics [176].

Limits	**Worldvolume formalisms** x^μ=operators	**Target space formalisms** x^μ=label
	Unknown M-theory formalisms	
Compactified on S^1	✓ Worldsheet string theory $X^\mu : \mathbb{C} \to \mathbb{R}^{9,1}$	SFT $\phi : \mathbb{R}^{9,1} \to$ Operators acting on string Fock space
$l_s \to 0$	Worldline formalism $x^\mu : \mathbb{R} \to \mathbb{R}^{n-1,1}$	QFT $\phi : \mathbb{R}^{n-1,1} \to$ Operators acting on Fock space ✓
Fock → Hilbert	Heisenberg picture $x^i : \mathbb{R} \to \mathbb{R}^{n-1,1}$	Schrödinger picture $\psi : \mathbb{R}^{n-1,1} \to \mathbb{C}$ ✓
$c \to \infty$	Heisenberg picture $x^i : \mathbb{R} \to \mathbb{R}^n$	Schrödinger picture $\psi : \mathbb{R}^n \to \mathbb{C}$ ✓
$\hbar \to 0$	✓ Newtonian mechanics $x^i : \mathbb{R} \to \mathbb{R}^n$	Koopman–von Neumann mechanics $\psi : \mathbb{R}^n \to \mathbb{C}$

Table 1. Summary of fundamental physics formalisms. Note that the "Fock → Hilbert" limit on the target space that gives *Relativistic Quantum Mechanics* can also be stated as the **classical worldline gravity limit** (matter on the worldline is still quantum). If we consider quantum gravity on the worldline, we need to sum over different worldlines, which gives rise to QFT (multiple particles) on the target space. Each worldline is nothing but a Feynman diagram. I didn't include classical field theory in the table, but they, too, have Feynman diagrams; see [231]. It is often said that only tree-level diagrams contribute to classical field theory, but there are exceptions; see [435]. If we only quantize worldline gravity, we get classical field theory; if we only quantize worldline matter, we get worldline quantum mechanics. Only when we quantize both do we get QFT. ✓ tells which of the 2 formalisms is popular.

The above definition is given to resolve the common confusion about whether the 3 Newton's laws of motion are just definitions or empirical facts [177–180] related to inertial frames. Newton's laws are usually stated as

1. A body remains at rest, or in motion at a constant speed in a straight line, except insofar as it is acted upon by a force.

2. The net force on a body is equal to the body's instantaneous acceleration multiplied by its instantaneous mass or, equivalently, the rate at which the body's momentum changes with time.

3. If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

The above traditional versions are bad. Because the 2nd law can be considered as 1) the definition of force $\mathbf{F}(\mathbf{x}(t)) = m\frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2}$ or 2) the definition of inertial frames or 3) some empirical fact about the nature of forces. The 1st law is just a corollary of the 2nd when the net force is $\mathbf{0}$. The 3rd law says two things: that the forces are instantaneous, and total momentum is conserved for isolated systems. In classical field theories like electromagnetism, due to the force field *not* propagating instantaneously, the 3rd law is not true even though momentum is still conserved. The below are better versions;

1. Inertial reference frames do exist. To find these, we need to see if an isolated particle is accelerating or not.

2. Forces (quantification of how much interaction there is) on particles are **empirically** observed to behave like mathematical vectors, and the acceleration of a particle is

proportional to the net vector addition of all forces, and mass is defined as the inverse of the proportionality constant.

3. Forces are **empirically** observed to be (almost) instantaneous. So, we can assume they are instantaneous in our theory. It is also **empirically** observed that the laws of nature have translation symmetry, and these laws apply to any point in our space. So, according to Noether's theorem, the total momentum is conserved.

This 1st law is saying that there exists an inertial frame. Once you believe in that, you can do experiments in any random reference frame, and as long as you carefully add fictitious forces like Coriolis force, centrifugal force, etc., Newton's laws will work. This 2nd law is an empirical fact that force behaves like a vector and not like some other thing like **scalar** or **pseudo-vector** or **spinor**, etc. It is also an empirical fact that it is proportional to **acceleration** and not $\frac{\mathrm{d}^n\mathbf{x}(t)}{\mathrm{d}t^n}$ for some $n > 2$. This is why you ***only need the positions and velocities*** of all Newtonian particles for complete knowledge to predict everything. Otherwise, we would need more input data, such as acceleration, jerk, and so on.

Coriolis force:

3.1.1 Newtonian gravity

Check section 4.2.1 for technical treatment of Newtonian gravity as a classical field theory.

Newton's law of universal gravitation:

Theorem 8 (Bertrand's theorem).

Two-body problem (Kepler problem):

Laplace–Runge–Lenz vector

Three-body problem:

3.2 Lagrangian formulation

Remark 11 (Galilean invariant Lagrangians).

$$\begin{aligned} t &\longrightarrow t, \\ \mathbf{r}_i &\longrightarrow \mathbf{r}_i - \mathbf{v}t, \end{aligned} \tag{3.2.1}$$

$$\begin{aligned} L_1 &= \sum_{i=1}^{N} \frac{m_i}{2}(\dot{\mathbf{r}}_i - \mathbf{u})^2 + \lambda \cdot \dot{\mathbf{u}} - V, \\ V &:= \sum_{1\le i<j\le N} V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|), \end{aligned} \tag{3.2.2}$$

$$\begin{aligned} t &\longrightarrow t, \\ \mathbf{r}_i &\longrightarrow \mathbf{r}_i - \mathbf{v}t, \\ \mathbf{u} &\longrightarrow \mathbf{u} - \mathbf{v}, \\ \lambda &\longrightarrow \lambda. \end{aligned} \tag{3.2.3}$$

Now we can integrate out the Lagrange multiplier λ to get

$$L_2 = \sum_{i=1}^{N} \frac{m_i}{2} (\dot{\mathbf{r}}_i - \mathbf{u}_0)^2 - V \tag{3.2.4}$$

$$\begin{aligned} t &\longrightarrow t, \\ \mathbf{r}_i &\longrightarrow \mathbf{r}_i - \mathbf{v}t, \\ \mathbf{u}_0 &\longrightarrow \mathbf{u}_0 - \mathbf{v}. \end{aligned} \tag{3.2.5}$$

3.2.1 Noether's theorem

3.3 Hamiltonian formulation

Theorem 9 (Liouville's theorem).

Theorem 10 (Non-squeezing theorem & the symplectic camel).

Constraints:

3.4 Hamilton–Jacobi formulation

3.5 Koopman–von Neumann formulation

[181–186, 199]

Remark 12 (Feynman diagrams). [187]

3.6 Statistical thermodynamics

[188]

Note. Like bosons, many classical particles can also occupy a single state (in this case, position on the phase space).

Theorem 11 (H-theorem).

Theorem 12 (Bohr-Van Leeuwen theorem).

Remark 13 (Gibbs paradox).

3.7 Dynamical systems

[193–199]

3.7.1 Chaos

3.7.2 Integrability

[239, 240][358]

3.8 Special relativity

[200, 201] and chapter 12 of [233] and chapter 6 of [262].

Remark 14 (Random history). Einstein's contribution to special relativity is not as significant or remarkable as his contribution to general relativity. In his work on special relativity, he showed that he was a man with enough courage to question fundamental aspects of reality, such as simultaneity and the concept of time, but special relativity is not conceptually deep like general relativity. The mathematics he used for special relativity is very elementary. Lorentz transformations were already known. But people incorrectly interpreted them, using the aether medium before Einstein clarified the meaning of those equations. Initially, Einstein thought that the geometric interpretation introduced by his teacher Minkowski was unnecessarily complicated mathematics introduced into this theory he even said "Since the mathematicians have invaded the theory of relativity, I do not understand it myself any more". Only later did he realize the importance of mathematics (especially geometry) in his decade-long quest to formulate general relativity. The fact that Einstein was a remarkable genius is only clear from his contributions to general relativity. Unlike in special relativity, where **Lorentz, Poincare, Larmor, Voigt, FitzGerald and many others** contributed, Einstein ***almost*** single-handedly formulated general relativity (with David Hilbert being the 2nd most important contributor who came up with the correct Einstein's field equations 5 days before Einstein independently but Einstein's paper was published first. But Hilbert rightfully acknowledged Einstein as the main contributor to GR because Einstein previously found the equations with the trace term missing and was aware that he needed to add some term.)

Postulate: Space and time are unified to give the flat Minkowski spacetime, and we can gauge fix the diffeomorphism invariance of special relativity so that the metric will just become $(-1, +1, +1, +1)$[9] in the Cartesian coordinates. The spatial metric is $\eta_{ab} = (+1, +1, +1)$[10]

This postulate is better than the original 2 postulates by Einstein as it is easier to generalize to general relativity.

Remark 15 (Misconceptions). I will explain why the following 7 statements are wrong.

1. *The theory of relativity is a generalization of classical mechanics and thus requires more postulates.*
   While it is a generalization of Newtonian physics since we can get back Newton's laws as $c \to \infty$, the number of postulates actually decreases despite being a more general theory. In Special Relativity, we have a unique metric connection/Christoffel symbols for a given metric, such as the Rindler metric. But in Newtonian mechanics, we have 2 metrics, 1 spatial and 1 temporal. When we have a coordinate system that is accelerating at constant $\vec{a}$ and another that is inertial, then the former has non-trivial metric connection/Christoffel symbols (they are nothing but pseudo forces/fictitious forces) even though both have the metric $(+1, +1, +1)$. People often think we are introducing one more postulate, the constancy of the speed of light.

2. *Special relativity has something to do with light (electromagnetic radiation) and the constancy of the speed of light.*
   Even when a universe has no electromagnetic field, Special Relativity still makes sense. For example, imagine a universe that only has a massive relativistic scalar field. Special Relativity is merely the kinematical laws of the universe, and this is unrelated to any interaction, such as electromagnetic radiation.

3. *Fast moving objects appear contracted in the direction of their motion.*
   It's not appear, it's literal, and there won't be any stress/strain, etc.

4. *To describe non-inertial reference frames, one must use the General Relativity.*
   No. Example: Rindler metric etc.

5. *To explain the twin paradox, one must introduce non-inertial reference frames.*
   While the brother who returned is obviously a non-inertial observer, we can do it completely from the point of view of the inertial brother.

6. *The faster a body moves, the larger its mass becomes.*
   The relativistic mass concept is now outdated. Mass now only means invariant mass.

[9] The other signature is considered blasphemy against the laws of physics.
[10] Latin indices are reserved for Euclidean signature. Greek indices are reserved for Minkowski signature.

7. *Time and space on equal footing in special relativity.*
Due to the thermodynamic arrow of time, they aren't.

Definition 18 (Gyrogroup). [202]

Lagrangian formulation:
Hamiltonian formulation:
Accelerating reference frames:

Theorem 13 (Thomas precession).

3.9 Spin λ particles with Maxwell–Boltzmann statistics

3.9.1 Newtonian

Chapter 3 of [207] and [203–205] and and section 3.3 in [206] from page 134.

3.9.2 Relativistic

Chapter 5 of [208] and [209–212]

4 Classical field theory

[213–222] and chapter 1 of [223] and chapter 5 of [61]
In classical physics, the matter is generally point particles and not fields, but the forces between them are generally classical fields. We have fields with different spins like Newtonian gravity is a *non-relativistic scalar field*, electromagnetism is a *relativistic vector or gauge field*, and Einstein's GR is a *relativistic spin 2 field*.

We can also study spin 1/2 fields as classical fields with Grassmannian values, but they are not useful for describing classical macroscopic physics and their only usefulness is to quantize them to get quantum fermion fields. Just like the ϕ^4 theory in QFT, these classical spinor fields are toy models. **The real reason** for this is that in our universe, fermions like electrons are charged, so if you want to have a large number of them to create a classical electron field, you can't do it due to the large repulsions. In some books, it's said that the reason we see classical bosonic fields like electromagnetism, and Newtonian gravity in ***day-to-day life*** but not fermionic fields is the Pauli exclusion principle. I don't think that is a correct reason because, in the classical limit, these electrons behave like point particles and if we **remove the coupling** to the electromagnetic field then there won't be repulsions and we can keep these electrons on a lattice and each of these electrons will have different state due to the different position on the lattice. When we take the lattice limit to zero, we get an effective classical spin 1/2 field that follows Maxwell–Boltzmann statistics (since $\hbar/2 \to 0$). At the end of the day, a quantum spin 1/2 field is nothing but the sum of different classical spin 1/2 field configurations in the path integral formalism. If the latter doesn't make sense, then the former also can't.

> **Remark 16.** Any molecule that has an odd number of nucleons is a fermion, and for a large enough molecule, we can neglect the spin $\hbar/2$, and it behaves like a particle. But fundamentally, it's a fermion, so it will have Grassmannian degrees of freedom just like the electron field in QED, but just like all experimentally observable things of QED are real numbers, it will still have real number experimental data [436], so at the end of the data you can never measure a Grassmann number and we can only measure the bosonic numbers. See 10.4.2 in [350], where they studied a fluid made up of a classical spinor field, but once we integrate out the unobservable Grassmann degrees of freedom, they get a fluid which is just a classical bosonic field.

4.1 Formalisms

4.1.1 Lagrangian field theory

4.1.1.1 Noether's theorem

4.1.2 Hamiltonian field theory

4.1.2.1 Ostrogradsky's ghosts

[224–226]

4.1.3 De Donder–Weyl theory

4.2 0 : Scalar fields

4.2.1 Newton–Cartan reformulation of Newtonian gravity

[227–229]. Check section 3.1.1 for the basics of Newtonian gravity.

$$\boxed{\text{Newtonian gravity} + \text{diffeomorphism invariance(gauge redundancy)} = \text{Newton–Cartan theory}} \tag{4.2.1}$$

Newtonian gravity is the first classical field theory to be discovered, and it is a nonrelativistic scalar field theory. We can add diffeomorphism invariance to Newtonian gravity to obtain the so-called Newton–Cartan theory. Since diffeomorphism invariance is nothing but a gauge redundancy, the theory we obtain is identical to Newtonian gravity.

4.2.2 Relativistic scalar field

> **Remark 17** (Feynman diagrams)**.** [230, 231]

4.2.3 Nordström's scalar gravity

Classical Feynman diagrams:

4.3 $1/2$: Classical spinor or Grassmann fields

[213, 214] and also see [232] for curved spacetime.

4.4 1 : Electromagnetism

[233, 234][215, 219]

4.4.1 Basics

4.4.2 As a $U(1)$ gauge field

4.4.3 In differential forms language

4.4.4 The energy-momentum tensor

4.4.5 Electromagnetic waves

4.4.6 Galilean electromagnetism ($c \to \infty$)

[235]

4.5 1 : Yang-Mills theory

[213–221] all discuss Yang-Mills. See [236] for Koopman–von Neumann formalism.

> **Remark 18** (Dressing Field Method)**.** [237, 556]

4.6 1 : Fluids

4.6.1 Non-relativistic fluids

[238]

4.6.2 Relativistic fluids

[242] and chapter 10 of [350].

4.7 Spontaneous symmetry breaking

4.8 Dynamical systems

4.8.1 Integrability

[239, 240][215]

4.8.2 Solitons

[215]

4.9 Classical statistical field theory

[241]

5 General relativity

[242–262]

In principle, this is also a classical field theory, and this section should be a subsection of the previous section. But in some sense, even **classical gravity is secretly already a quantum theory** since even in the classical limit, gravity is dual to some holographic quantum field theory. Apart from that, I really like general relativity (even more than the standard model of particle physics since it has no dimensionless parameters and can be completely guessed by anyone who knows Newtonian gravity and Maxwell's equations without experimental help), so it deserved its own section.

> General relativity is the greatest feat of human thinking about nature, the most amazing combination of **philosophical** penetration, **physical** intuition, and **mathematical** skill.
>
> *Max Born*

> The theory of gravitational fields, constructed on the basis of the theory of relativity, is called the general theory of relativity. It was established by Einstein (and finally formulated by him in 1915), and represents probably the most beautiful of all existing physical theories. It is remarkable that it was developed by Einstein in a **purely deductive manner** and only later was substantiated by astronomical observations.
>
> *Lev Landau*

> There was difficulty reconciling the Newtonian theory of gravitation with its instantaneous propagation of forces with the requirements of special relativity; and Einstein working on this difficulty was led to a generalization of his relativity—which was probably the greatest scientific discovery that was ever made.
>
> *P.A.M Dirac*

5.1 Introduction

Before discussing how to formulate general relativity, let's discuss what "general covariance" means. Almost everywhere, people mention that General Relatively & its generalizations uniquely have general covariance. But, **EVERY** physical theory can be formulated in a generally covariant way, and it is not a specialty of general relativity. Long after writing the below subsection based on my understanding, I found that section 7.1 of [253] and section 4.2 of [537] also explain general covariance well and agree with what I wrote. Apparently, this confusion was pointed out to Einstein as early as 1917 by Kretschmann; see [264] for the history. See also [265, 266]. In the next section, I will explain these 4:

1. General covariance is a gauge/fake symmetry present in all physical theories.

2. "No prior geometry", also called Background independence, is special to relativistic[11] gravity theories (both classical and quantum).

3. Local Lorentz invariance follows directly from the definition of a pseudo-Riemannian manifold, just like Riemannian manifolds are locally Euclidean, and it is not a gauge symmetry, and it is there in any relativistic theory, including special relativity, electromagnetism, etc.

4. When we consider diffeomorphism invariance as a gauge symmetry of GR, it ***doesn't have any real global symmetry that is associated*** with it. For example, in electromagnetism, the gauge $U(1)$ symmetry contains the global $U(1)$ symmetry as a subset. Only this global $U(1)$ gives the charge conservation; the local $U(1)$ part is fake and doesn't give any nontrivial conserved quantities. In the context of symmetry breaking, recall Elitzur's theorem 7.9.2 that only the global part can be broken as it is real, and the gauge part can't be broken as it is fake. But in the case of diffeomorphism, even the global part is a fake symmetry that can't be broken, for example, consider a coordinate transformation $x'^{\mu} = x^{\mu} + a^{\mu}$, so $g'_{\mu\nu}(x') = g_{\rho\sigma}(x)\frac{\partial x^{\rho}}{dx'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} = g_{\rho\sigma}(x' - a)$. This is present in all spacetimes & is not a real global translation symmetry, unlike the real global symmetries present in Minkowski/AdS/dS, etc, which need Killing vector fields.

5.1.1 General covariance vs No prior geometry vs Local Lorentz invariance

> Mathematics was not sufficiently refined in 1917 to cleave apart the demands for "**no prior geometry**" and for a geometric, coordinate-independent formulation of physics. Einstein described both demands by a single phrase, "general covariance". The "no prior geometry" demand actually fathered general relativity, but by doing so anonymously, disguised as "general covariance", it also fathered half a century of confusion.
>
> *MTW Gravitation* (1973 book)

General covariance or **diffeomorphism invariance** or **reparameterization invariance**: The laws of physics will be invariant under arbitrary differentiable coordinate transformations.

This should have been understood long before general relativity, as it is true for all physical theories, but this was understood by Einstein while he was developing general relativity and caused a lot of confusion because Einstein thought this was unique to his general relativity. Recall Newton's second law

$$\mathbf{F}(\mathbf{x}(t)) = m\frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2}$$

The above equation is valid for any coordinate system. We generally use Cartesian coordinates because they are the simplest. But if you use spherical coordinates, then the components of the above equation will be

[11]Perhaps it should be called Lorentzian as relativistic might also mean Galilean relativistic.

$$F_r = m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta)$$
$$F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta)$$
$$F_\phi = m(2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)$$

Even though the equations look very different, the physics hasn't changed. The reason things became more complicated is because the metric went from diag$(1,1,1)$ to something slightly more complicated. Similarly, we can go to an arbitrary coordinate system with arbitrary metric g_{ab} that is more complicated than spherical coordinates, and Newton's second law is still valid. **We can use a lot of machinery of Riemannian geometry in Newtonian physics**[12] as shown in 4.2.1. But in general, it is not needed because, in this case, the background space is not dynamic, and geometry is *a priory* fixed. We have a flat $3D$ Euclidean space, i.e., scalar curvature is 0 everywhere. So we can always choose (i.e., **globally gauge fixing**) the global Cartesian coordinates in this case where the metric is simplified to diag$(1,1,1)$. If our space (non-dynamic) instead has some arbitrary curvature, then we can't find nice coordinate systems like Cartesian coordinates, and we should work with nontrivial metrics and use the machinery of Riemannian geometry. **Example**: Think about particles confined to a spherical surface interacting with Newtonian gravity. F_r component will be cancelled due to normal confinement forces and $r = const$. This submanifold of $3D$ Euclidean space is a $2D$ **non-dynamic** curved space. Roughly speaking, what we call Newtonian gravity is a gauge-fixed version of Newton–Cartan theory 4.2.1.

This logic can be carried over to quantum mechanics, special relativity, quantum field theory etc. All of them **must** have diffeomorphism invariance. You probably saw $\vec{\nabla}$ in cartesian, spherical, and cylindrical coordinates in Quantum Mechanics. But we can also define an arbitrary coordinate system on $3D$ Euclidean space with some arbitrary metric that will still have 0 scalar curvature everywhere. In special relativity and quantum field theory, we have different coordinates that are not Cartesian, such as the Rindler coordinates[13]. In principle, we can take an arbitrary coordinate system where the metric is very different from diag$(-1,1,1,1)$. Note that QFT in curved spacetime $\neq$ Generally covariant QFT, because in the former, we have an arbitrary non-dynamical spacetime, but in the latter, we have the specific Minkowski spacetime, even though in both cases the coordinate system is arbitrary.

Active and passive transformation: Mathematically they are same. The mathematical equation for your rotation by an angle is the same as if everything else in the universe revolves around you by the same angle. Whether they are physically the same is an **ongoing** philosophical debate, see section 8.2.2.3. Mach's principle states that the existence of absolute rotation (the distinction of local inertial frames vs. rotating reference frames) is determined by the large-scale distribution of matter in the universe. Though it motivated

[12] After looking for a lot of books, I found chapter 5 of [164], which studies Newtonian physics on arbitrary Riemannian manifolds and chapter 5 of [214] which studies Newtonian physics on fiber bundles. One might say these are nothing but uselessly flexing mathematical muscles or even like using a lawn mover to shave a beard, but the main point is that general covariance is there in every physical theory.

[13] Recall Unruh effect.

Einstein to come up with general relativity, it is not exactly known which form of Mach's principle is valid, and theories like Brans–Dicke theory obey a stronger form of Mach's principle than GR.

Diffeomorphism invariance is a gauge symmetry: Recall that only the global part of a gauge symmetry is physical and gives rise to **conserved quantities**. Diffeomorphism invariance is present in any theory. For example, if you consider special relativity and gauge fix the metric so that it becomes $\mathrm{diag}(-1, 1, 1, 1)$, then there is still a global symmetry left corresponding to the Poincaré group. Spacetime translations give 4-momentum conservation. Spacetime rotations give angular momentum conservation (due to spatial rotations) and "conservation of the center of mass" also called the conservation of $\mathbf{N} = t\mathbf{p} - E\mathbf{r}$ (due to Lorentz boosts). General relativity, in general, has fewer global symmetries than special relativity, so we get fewer conserved quantities.

Local Lorentz invariance is not a gauge symmetry: This is not a gauge/fake symmetry because this has physical consequences such as causal structure and is **not a mere redundancy**. By definition, a Lorentzian manifold (a subset of pseudo-Riemannian manifolds) should locally look like $\mathbb{R}^{n-1,1}$. Despite common confusion, this is ***unrelated*** to general covariance (diffeomorphism invariance). Newtonian physics happens on manifolds (i.e. locally Euclidean $\mathbb{R}^n$) and they still have general covariance. There are various generalizations of manifolds called generalized smooth spaces [267] with orbifolds being an example. It doesn't matter if you use manifolds or Lorentzian manifolds or complex manifolds, or even generalizations like orbifolds, general covariance (diffeomorphism invariance) will always be there, and it's even more fundamental than these locally smooth structures. Note the obvious fact that local Lorentz invariance **doesn't imply** global Lorentz invariance, for example, the **FLRW metric** doesn't have time translation symmetry, unlike the Minkowski space.

No prior geometry or Background independence: This is the main specialty of general relativity compared to previous theories. Because the background is not a priori fixed to Euclidean or Minkowski space and because the background is dynamic, it becomes **absolutely necessary** to use the machinery of Riemannian geometry. For Newtonian gravity, it was merely optional.

> **Remark 19 (Abstract index notation or Coordinate-free approach or Index-free notation).** See the beginning of 2.1.5 about Descartes believing in the conspiracy that ancient geometers knew coordinate geometry but proved things using more complicated coordinate-free geometry to make mathematics prestigious. A point is not diffeomorphism invariant, but lengths, areas, etc, are diffeomorphism invariant. In the case of General Relativity, the coordinate approach was first discovered. But there exists a coordinate-free approach, see 5.2.4, using differential forms, etc., where, for

example, the Einstein Field Equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$ look like

$$G + \Lambda g = \kappa\, T \tag{5.1.1}$$

5.1.2 Equivalence principle

5.1.3 Einstein field equations

5.2 Formalisms

[263]

The previous section was about guessing the correct gravitational equations of motion by conceptual arguments. This was the original historical way Einstein developed it. But this section discusses some more formal versions of General Relativity. But I would like to mention that some formalisms, such as the ones below, are covered in my QG notes.

1. Worldline formalism of GR is in section 3 of QG notes
2. The twistor formalism, known so far only for self-dual space-times, is equivalent to GR. It is in section 14 of QG notes.
3. The Hamiltonian formalism using Ashtekar variables is in Loop Quantum Gravity part of QG notes.

5.2.1 Einstein–Hilbert action

5.2.2 Dirac Hamiltonian and ADM Hamiltonian

[268]

5.2.3 BMS formalism

[269, 270]

5.2.4 Coordinate-free approach

[271]

See Remark 19. In this section, we work with only diffeomorphism-invariant objects, so we don't introduce fake/gauge degrees of freedom.

5.2.5 Vielbein formalism

5.2.6 By gauging

[272]

5.2.7 MacDowell-Mansouri formalism

[273]

5.2.8 As a spin-2 theory

[274, 275]

5.2.9 2-Spinor formalism

[276] and also the famous volumes of Penrose and Rindler, and chapters 3 and 4 of [277].

5.2.10 Einstein algebras

[278–281]

5.2.11 Axiomatic Ehlers-Pirani-Schild formalism

[282, 283]

5.3 Exact solutions

[284]

5.3.1 Petrov classification

5.4 Black holes

5.4.1 Schwarzschild metric

5.4.2 Reissner-Nordström metric

5.4.3 Kerr-Newman metric

[285]

5.4.3.1 Penrose process

5.4.4 The Four Laws

5.4.5 Higher D

[286, 287]

5.4.6 Regular black holes

[288, 289]

5.5 Causal structure

[290, 291]

5.5.1 Singularity theorems

[292]

Definition 19 (Singularity)**.** [293]

5.5.2 FTL and time travel

[294–297]

5.6 Perturbation theory

5.6.1 GR→NG

The limit to get special relativity is very obvious. The metric just becomes non-dynamical and becomes the Minkowski metric. The limit to get Newtonian Gravity is nontrivial. The following passage from 2.1.4 of [299] explains how even the leading order theory already differs from Newtonian gravity.

"Therefore, general relativity produces the same trajectories at leading order as Newton's theory. This is called the Newtonian approximation of general relativity.

However, let us stress that even at this level of approximation, the two theories differ drastically—in a way that can be tested at the experimental level already. Indeed, in general relativity, the variation of an observer proper time $d\tau$ (Eq. 27) with respect to the proper time of another observer depends explicitly on their different positions in a gravitational potential U . This means that two observers at different locations in the gravitational potential will not agree on the evolution of time. This effect, although minute, can be tested if one has accurate enough clocks. In other words, had we developed atomic clocks with sufficient precision prior to our ability to observe the motions of celestial bodies in the solar system, we could have confirmed the superiority of general relativity over Newton's theory."

5.6.2 Post-Newtonian expansion

5.6.3 Minkowskian and post-Minkowskian approximation

5.6.4 Gravitational waves

[301, 302]

5.7 The Cauchy problem

[242, 303]

5.8 Statistical thermodynamics

Chapter X of [242] and [305–307].

5.8.1 Ehrenfest–Tolman effect

5.9 Cosmology

[310–313]

5.9.1 de Sitter space

[314]

> **Remark 20** (Translation symmetry). You might think dS first contracts and then expands, so how can it have the full translation symmetry like the Minkowski spacetime, but this is merely because we are using certain coordinates. It's like in a 2D spherical surface, the latitudes increase and then decrease, but it still has as much symmetry

as $\mathbb{R}^2$. As explained in Remark 7, whether something is an intrinsic translation or an intrinsic rotation is dependent on the observer or coordinates.

5.9.1.1 dS-Schwarzschild metric

5.9.2 Anti-de Sitter space

5.9.2.1 AdS-Schwarzschild metric

5.9.3 FLRW metric

5.9.4 The inhomogeneous universe

[310]

5.9.4.1 Newtonian perturbation theory

5.9.4.2 Relativistic perturbation theory

5.9.4.3 Cosmic microwave background

5.9.5 The standard model of cosmology (ΛCDM)

Note: There is a difference between how GR is related to ΛCDM compared to how QFT is related to the standard model of particle physics; see the table in 7.14. GR is already a single theory. But QFT is a framework that can describe ∞ theories. **Horndeski's theory** [318] is more analogous to QFT than GR. Horndeski's theory is more like a framework for classical gravity theories, and GR is just one of them. But there is a **uniqueness** to GR. GR is the **simplest classical gravity theory** and is preferred by the Occam's razor. In QFT, we don't have this kind of uniqueness. Maybe you can argue that the gauge group $U(1) \times SU(2) \times SU(3)$ is somehow unique, but even then, the dimensionless parameters coming from masses, etc, are arbitrary experimental values. ΛCDM itself is **not a theory** but just a single solution to GR that describes the real universe and is dependent on the initial conditions at the initial singularity. ΛCDM is a *course-grained approximation of the universe*. It explains the large-scale structure of the universe, but it is not an exact model that contains the metric at each point in the universe. Other solutions to GR might also exist in the multiverse. It is possible that ΛCDM is a uniquely preferred solution when we consider Quantum Gravity (maybe just like electron $g-2$, we can precisely calculate the parameters of ΛCDM), but in GR, it is not the case.

5.9.6 Inflation

[315]

5.9.7 Modified gravity

[316–318]

5.9.7.1 Horndeski's theory
[318]

5.9.8 Ultimate fate of the universe scenarios

Part II

Quantum physics

6 Quantum mechanics

[319–327]. For recent non-perturbative developments, see this book [328].

6.1 Formulation

6.1.1 Canonical formulation

Definition 20 (Dirac bracket).

Note. An operator expressed in some basis is not the same as that operator itself.

$$\langle x|\hat{p} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\langle x| \;\not\Rightarrow\; \hat{p} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} \text{ and } \langle p|\hat{x} = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p}\langle p| \;\not\Rightarrow\; \hat{x} = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p} \tag{6.1.1}$$

Example 12 (Hydrogen atom).

6.1.2 Path integral formulation

Remark 21 (Hamiltonian path integrals). Path integrals are not unique to the Lagrangian formalism & can be used for the Hamiltonian mechanics also.

Example 13 (Hydrogen atom from Path Integrals). [331–333]

6.2 Rotations and angular momentum

6.3 Perturbation theory

6.4 QM→CM

References: Chapter 14 of [319] and [334, 335].

Spin: Note that $U(\vec{\theta}) = e^{i\vec{\theta}\cdot\vec{S}/\hbar} = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}$ is independent of $\hbar$. But $\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{L} + \hbar\frac{\vec{\sigma}}{2}$. is dependent on $\hbar$ and in the limit $\hbar \to 0$ the internal component of the angular momentum disappears. So, in essence, the spin internal degrees of freedom decouple from the classical angular momentum. But they are still there. So, there are still intrinsic spinor degrees of freedom defined by the rotations $U(\vec{\theta})$ but now these degrees of freedom are decoupled from $\mathbf{J} = \mathbf{L}$.

6.4.1 Decoherence

[336, 337]

6.5 Axiomatic QM

[338–343]

6.5.1 Dirac–von Neumann axioms and C*-algebras

6.5.2 Spectral theory

6.5.2.1 Rigged Hilbert spaces

[344]

6.5.3 Geometric quantization

[347–355]

6.5.4 Deformation quantization

[356]

6.6 Quantum information theory

[357]

6.7 Dynamical systems

6.7.1 Integrability

[358, 359]

6.7.2 Chaos

[360, 361]

6.8 Statistical thermodynamics

6.8.1 Bose-Einstein statistics

6.8.2 Fermi-Dirac statistics

6.9 Nonrelativistic particle with spin j

[362][433]

6.10 Relativistic quantum mechanics (RQM)

[363–365, 508] and chapter 1 of [398]

Definition 21 (Pauli-Lubanski pseudovector).

6.10.1 0 : Klein–Gordon equation

6.10.2 $1/2$: Dirac, Weyl and Majorana equations

[366]

Hydrogen Atom:

$$E_{j\,n} = -m_{\mathrm{e}}c^2 \left[1 - \left(1 + \left[\frac{\alpha}{n - j - \frac{1}{2} + \sqrt{\left(j + \frac{1}{2}\right)^2 - \alpha^2}}\right]^2\right)^{-\frac{1}{2}}\right]. \tag{6.10.1}$$

6.10.3 1 : Maxwell equation and Proca equation

6.10.4 $3/2$: Rarita–Schwinger equation

6.10.5 j : Bargmann–Wigner equation and Joos–Weinberg equation

7 Quantum field theory

[367–391][59–71, 74]. For non-perturbative QFT check these books [392–398]. [398][59] are the best books for conceptual depth. Most of these references use the $(+---)$ convention, which is blasphemy against gravity. But a few use the correct convention like [367, 379–381] (also Weinberg's volumes).

> The wave fields ϕ, φ, etc, are not probability amplitudes at all, but operators which create or destroy particles in the various normal modes. It would be a good thing if the misleading expression 'second quantization' were permanently retired.
>
> *Steven Weinberg (Vol 1)*

Quantum field theory is a misnomer. It's a framework and not a "theory", just like classical field theory is a framework.

The modern point of view is that we start with local unitary quantum fields defined by a Lagrangian to match the experimental data from accelerators. There might be a more fundamental theory, like string theory, that can explain the standard model. But for now, we assume QFT or, more precisely, the Standard Model as an axiom.

That being said, we have various theoretical and experimental motivations for why we should start with the QFT framework as an axiom. Below, I briefly review the motivations, but for a more in-depth explanation of how we can narrow down to the framework of QFT, check either Weinberg Vol 1 or chapters 5 and 6 of [59].

1. **Multi particle theory**: Relativistic Quantum Mechanics 6.10 is not incorrect, but it is a single-particle theory that is only[14] valid up to some energies. When the relativistic corrections are small, RQM is applicable, for example, to explain the fine structure, see (6.10.1). But RQM already indirectly via the Dirac sea taught us about the existence of antimatter. So, this means that when the energies are high enough, there will be virtual particles created by pair production. So, we can no longer have a theory that has a fixed number of particles. To explain things like Lamb shift, we need to upgrade to a multi-particle theory.
2. **Clustering and the smoothness of scattering amplitudes**:
3. **Local fields and non-localizable particles**:
4. **Unitarity**:

[14] As discussed later, renormalization group flows teach us that even the standard model is just an EFT that's only valid up to some energy scale.

7.1 0 : Scalar fields

7.1.1 Canonical quantization

7.1.2 Path integral quantization

7.1.3 Self-interacting scalar field theory

Failures of the Feynman diagram approach

1. It fails when $\lambda \sim 1$
2. It fails for complicated interaction terms like $sin(\phi)$.
3. It fails for bound states, even for simple interactions & weak coupling limit. For example, the hydrogen atom or positronium in QED.

Example 14 (Derivative interactions).

7.2 1/2 : Fermion fields

7.3 1 : Gauge fields

[404]

7.3.1 QED

Theorem 14 (Furry's theorem).

Landau pole:

7.3.2 Yang–Mills theory

> It happened that one semester [around 1970] I was teaching GR, and I noticed that the formula in gauge theory for the field strength and the formula in Riemannian geometry for the Riemann tensor are not just similar – they are, in fact, the same if one makes the right identification of symbols! It is hard to describe the thrill I felt at understanding this point...
>
> *Yang Chen-Ning*

7.3.3 BRST formalism

[406–411] and also chapters 45 to 47 of [367].

Gribov Ambiguity: [412, 413] and also [30, 369]

7.4 Discrete symmetries, continuous symmetries and all that

Nmae is inspired by the book "PCT, spin and statistics, and all that".

7.4.1 $\mathcal{CPT}$ symmetry

Also called $\mathcal{CRT}$ symmetry.

7.4.2 Ward–Takahashi identity for continuous symmetries

7.4.3 Spin-statistics theorem

> ...we should modify the meaning of "understand", and at the same time reduce our expectations of any proof of the Spin-Statistics Theorem. What is proved - whether truly or not, whether optimally or not, in an acceptable logical sequence or not - is that the existing theory is consistent with the spin-statistics relation. What is not demonstrated is a reason for the spin-statistics relation.
> To belabor the point, it is difficult to imagine a fundamental mechanism for the Pauli Exclusion Principle - upon which all depends - which predicates it (looking ahead to the work of Hall and Wightman) upon the analyticity properties of vacuum expectation values of products of quantized field operators. Did God - for lack of a better word - build a series of failed worlds which sputtered and died, or exploded and disintegrated, before discovering the stabilizing effect of anticommutation relations for half-integral spin fields? Was this before or after imposing the requirements of Lorentz invariance? Are we the lucky winners of a Monte Carlo simulation in which every choice was tried and one survived?
> Must we reduce our demands on physics to require only consistency? Does an understanding of the "Why?" of the spin-statistics relation have no direct answer in physics? Or must physics be formulated to include it? The Pauli result (Pauli's proof) does not explain the spin-statistics relation and cannot. The Neuenschwanders and the Feynmans of the world must remain unsatisfied (Ch.20)...
>
> *Ian Duck, E. C. George Sudarshan* (1998 book)

7.5 Scattering amplitudes

[414–416]

7.5.1 Spinor helicity formalism

7.6 Bound states

[399–401]

7.7 Renormalization

[418–424]

Example 15 (Ideal gas).

Example 16 (Water).

Example 17 (Two coupled harmonic oscillators).

Definition 22 (UV fixed point).

Definition 23 (UV finiteness)**.** Note that UV fixed points are a subset of UV completions. But UV completions need not be QFTs.

7.7.1 QED

7.7.2 Yang–Mills theory

7.8 Effective field theory

[481–486]

7.9 Symmetry breaking

[425, 426]

7.9.1 Abelian Higgs mechanism

7.9.2 Elitzur's theorem

> The terms "gauge symmetry" and "gauge symmetry breaking" are two of the most misleading terms in theoretical physics.
>
> *Xiao-Gang Wen* (Source)

7.9.3 Renormalizablity

7.10 Schrödinger functional picture

[427] and Chapter 10 and 11 of [428] and [429].

7.11 Euclidean QFT

[430, 431] and chapter 5 of [398]

7.12 Nonrelativistic QFT (NQFT)

[433]

7.13 QFT→NQFT or ClFT or RQM or QM

References: Chapter 8 of [59], [434] and Chapter 6 of [71]. We will only be discussing these complications for free theories. We can't define single-particle states for interacting theories due to Haag's theorem 7.26.1.1. But in interacting theories, S matrix is well defined, and we can define single-particle states at asymptotic infinity.

QFT under various limits gives the following 4 theories: 1) Nonrelativistic QFT, 2) Classical field theory (called ClFT as CFT means Conformal Field Theory), 3) Relativistic quantum mechanics, and 4) Quantum mechanics. We now study what these precise limits are.

1. **Nonrelativistic QFT** ($c \to \infty$): The nonrelativistic limit of a Klein–Gordon ***field*** is not the Schrödinger equation but the Schrödinger field.

2. **Classical field theory** ($\hbar \to 0$):

   (a) **Loops**: It is often said that only tree-level diagrams contribute to classical field theory, but there are exceptions; see [435].

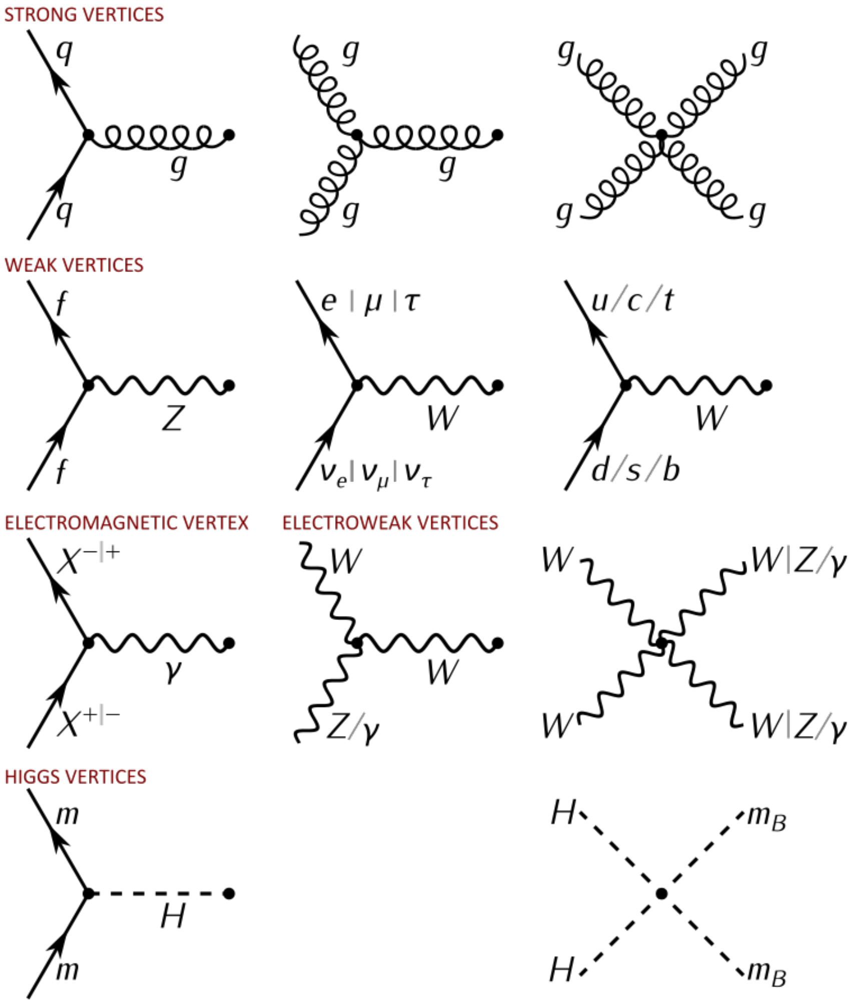
3. **Relativistic quantum mechanics** (Fock $\to$ Hilbert): Like in ordinary quantum mechanics, we consider the electromagnetic field as classical but the electron as a quantum ***relativistic*** particle. So, we are coupling a classical field with a fixed number of relativistic quantum particles.

4. **Quantum mechanics** ($c \to \infty$ and Fock $\to$ Hilbert): The nonrelativistic limit of a Klein–Gordon ***equation*** from Relativistic Quantum Mechanics is the Schrödinger equation not the Schrödinger field.

7.14 The standard model of particle physics

[437–441]
We should now touch some grass and make contact with reality. Using QFT (it's a framework, not a theory), you can make infinite theories, and the standard model is 1 of them that we can fix based on experiments. It needs 25 fundamental dimensionless constants determined by experiments. See the **Note** in 5.9.5.

Framework	Pseudo-Riemannian geometry or Horndeski's theory	QFT	String/M-theory
Theory	GR, Brans–Dicke theory, $f(R)$, and this list	Standard Model and any consistent EFT	String/M-theory (Unique)
Solutions	ΛCDM, Minkowski, dS, AdS, AdS-Schwarzschild	Scattering (In/Out) states, bound states	Compactifications/vacua with given matter fields
Observables	Geodesic lengths, areas, volumes, scalar curvature etc	In/Out states: Correlation functions or scattering amplitudes Bound states: Mass spectrum, binding energy, decay rates etc	AdS: Boundary correlation functions Flat: Scattering amplitudes



Remark 22 (Gauge group ambiguity). [442]

7.14.1 Electroweak theory

7.14.1.1 $V - A$ theory

[443]

7.14.1.2 Higgs mechanism

[444, 445]

7.14.2 QCD

[446–451]

7.14.2.1 Perturbative limit

7.14.2.2 Shifman-Vainshtein-Zakharov sum rules

[452]

7.14.2.3 Confinement

[453–460]. [460] is comprehensive.

7.15 BSM (Beyond the Standard Model)

[462]

7.15.1 Neutrino oscillations

[463]

7.15.2 Higgs physics

[444, 445]

7.15.3 Dark matter candidates

Chapter 10 of [464]

7.15.4 Baryon asymmetry

[465]

7.15.5 Grand unified theories

7.16 Lattice gauge theory

[466, 467]

7.17 Anomalies

[468, 469] and Chapters 13 and 17 of [369] and chapter 30 of [372].

7.18 Solitons

[470–472][392–395, 397]

7.18.1 Instantons

[473]

7.18.2 Monopoles

7.18.3 Domain Walls

7.18.4 Vortices

7.18.5 Flux Tubes

7.19 Dynamical systems

7.19.1 Integrability

[474]

7.19.2 Chaos

[475]

7.20 Statistical thermodynamics

[476]

7.20.1 Thermal QFT

7.20.2 Statistical field theory

7.21 QFT in lower dimensions

[477, 478]

7.21.1 0+0

7.21.2 0+1

It is often said that QFT is nothing but QM; this refers to the worldline formalism; check my QG notes.

It is also often said that the ordinary QM is nothing but a QFT. Let me clarify in what sense. In QM, if you take the position operator x, it is nothing but a map from the worldline to the target space, i.e., $x : \mathbb{R} \to \mathbb{R}^{n-1,1}$, which is nothing but the Heisenberg picture. Of course, we can also work completely on the target space by studying the wave function $\psi : \mathbb{R}^{n-1,1} \to \mathbb{C}$, which is nothing but the Schrödinger picture. Note that when we completely focus on the worldline, we get a theory living in a (0+1)-D spacetime. This (0+1)-D theory can be called QFT on the worldline but not QM. It is a QFT because x is an operator-valued function/distribution that fills the entire (0+1)-D spacetime (worldline).

As both the worldline formalisms of QM and QFT are (0+1)-D QFTs, they are important. We will now study (0+1)-D QFTs.

7.22 The large N limit

[396, 397][479, 480, 487–491]

7.22.1 $O(N)$ sigma model

7.22.2 Gross–Neveu model

7.22.3 $\mathbb{CP}^{N-1}$ model

Section 7.3 of [397].

7.22.4 QCD & colored theories

7.23 Resurgence

[492–497]

7.24 Entanglement

[498, 499]

7.25 Energy conditions

[500, 501]

7.26 Axiomatic quantum field theory

[502, 503]

7.26.1 Problems with nonrigorous QFT

7.26.1.1 Haag's theorem

[504–506]

7.26.2 Wightman axioms

[507–509]

7.26.3 Constructive QFT

[510–512]

7.26.4 Haag–Kastler Algebraic QFT

[513–515]

7.26.4.1 Bell's inequality

[517]

7.26.5 Topological QFT

[518–524]

7.26.5.1 Anyons

Chapter 22 of [378]

7.26.5.2 Topological defects

[525–527]

7.26.5.3 Functorial QFT

[528, 529]

8 Philosophy of physics

[535–538]

8.1 Philosophy of mathematics

[530]

8.1.1 Mathematical Platonism

[531]

8.1.1.1 Quine–Putnam indispensability argument

8.1.1.2 Physics without mathematics?

[532, 533]

8.1.2 Probability interpretations

[534]

8.2 Classical physics

Theory-ladenness:

Classical Mechanics:

> **Remark 23** (Coincidence)**.** How does the particle "know" the correct path in the Lagrangian formalism? The main difference between Newtonian formalism & Lagrangian formalism is that the former gives local boundary conditions (position & velocity) while the latter gives global boundary conditions. Some think that Lagrangian formalism only gives the Equations of Motion but doesn't give solutions [539], but that's wrong since it contains the same amount of physical information as the EoM.
>
> Is it just a coincidence that the particle always travels in the path with the least action? We have seen that coincidences in less fundamental theories are generally explained by deeper theories like GR, explaining why the equivalence of gravitational and inertial masses, and electromagnetism explains why the charge is the same in both electricity and magnetism. Similarly, quantum mechanics explains this coincidence by saying that a particle is not smart but very dumb and doesn't know the correct path; it just goes in every path possible. Miraculously, most of the paths cancel out at large scales, but the paths close to the classical add up to give the dominant contribution.

8.2.1 Thermodynamic paradoxes

[540]

Loschmidt's paradox (Arrow of time): [541, 542]

8.2.2 Spacetime

[543]

8.2.2.1 A/B-theory of time

8.2.2.2 Hole argument

[544, 545]

8.2.2.3 Mach's principle & Newton's bucket argument

[546]

8.2.3 Existence & uniqueness of initial value problems

Newtonian mechanics: Norton's dome is an example with non-unique solutions.
Navier–Stokes equations:
General relativity:

8.2.4 Cosmology

[547]

8.3 Quantum physics

> **Remark 24.** Is $\psi(x) \equiv \langle x|\psi\rangle$ more fundamental than $\psi(p) \equiv \langle p|\psi\rangle$ in quantum mechanics? Is this not true in the Hamiltonian formalism? Mathematically, they have the same information, but empirically, $\psi(x)$ has a better interpretation.

Hilbert spaces: [548]

8.3.1 Quantum interpretations

8.3.1.1 Bell's inequality

[549]

8.3.1.2 Constraints from relativity and QFT

[550–554]

8.3.2 QFT

[535–537, 551, 555, 556]

8.3.3 Gauge theory

[557–564]

8.3.3.1 Aharonov–Bohm effect [565]

8.3.4 The ontology of particles or fields

Part Nine of [536]. Also [566].

8.3.5 EFT

[567–570]

8.3.6 Quantum gravity

[571, 572]

A Failed theories

Some historically important failed theories.

1. Aristotelian physics (4th-century BC)
2. Brahmagupta's "gurutvākarṣaṇam" qualitative theory of gravity (628)
3. Descartes' vortices theory of gravity (1644)
4. Aether theories before special relativity (1704-1905)
5. Einstein's scalar field theory for gravity (1912)
6. Old semiclassical quantum theory (1900–1925)
7. Relativistic quantum mechanics (1927)
8. Steady-state model (1930s and 40s)
9. Einstein's classical unified theory (1930-1955)
10. Technicolor (1980s)
11. String theory (as an alternative to QCD it failed, as quantum gravity/theory of everything, it still is the most promising candidate)

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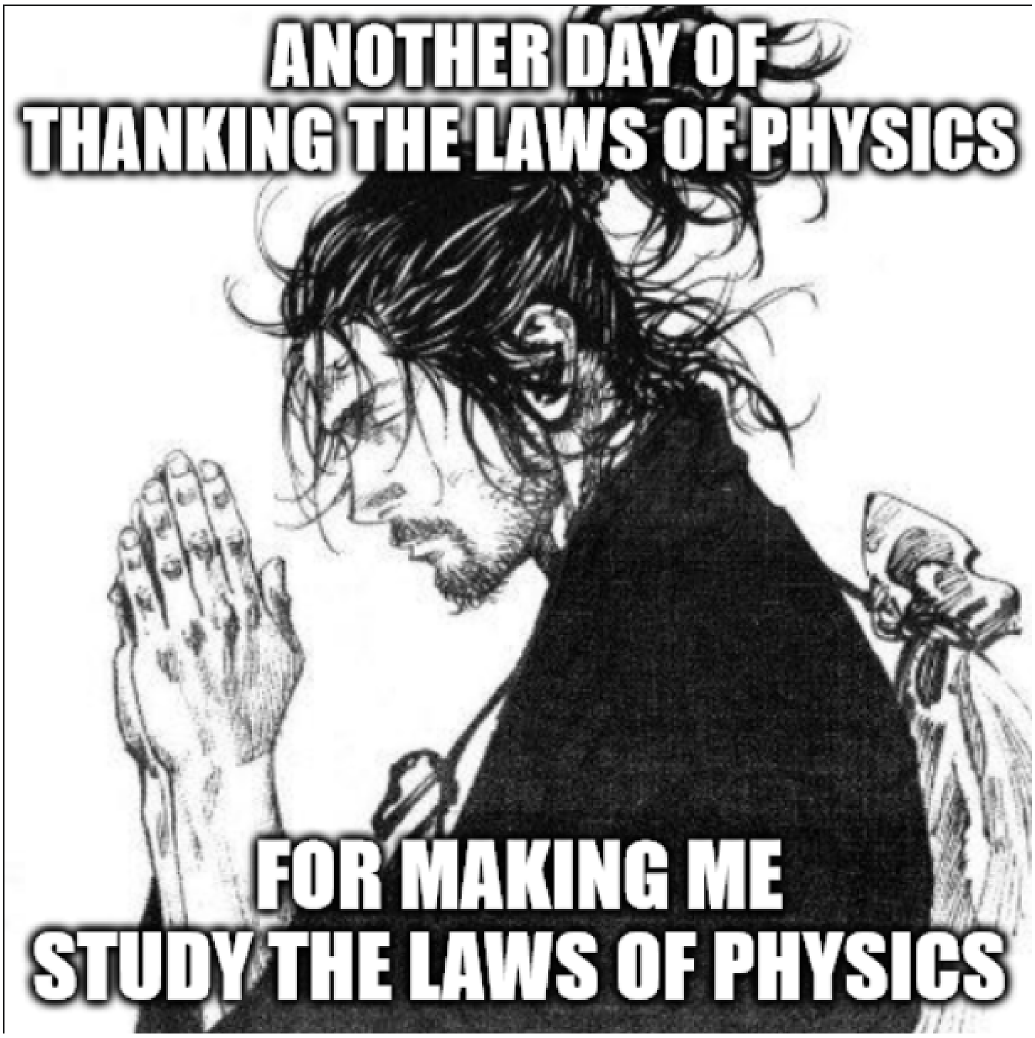
IN THIS WORLD, IS THE DESTINY OF MANKIND CONTROLLED BY SOME TRANSCENDENTAL ENTITY OR LAW......?
IS IT LIKE THE HAND OF GOD HOVERING ABOVE?
AT LEAST, IT IS TRUE THAT MAN HAS NO CONTROL, EVEN OVER HIS OWN WILL.
ベルセルク

Acknowledgments

I acknowledge that life is meaningless. I thank the fundamental laws of physics for being mathematically comprehensible and mysterious, but I will not thank the laws of physics for being amoral and absurdly pointless.



How strange is the lot of us mortals! Each of us is here for a brief sojourn; for what purpose he knows not, though he sometimes thinks he senses it. But without deeper reflection one knows from daily life that one exists for other people — first of all for those upon whose smiles and well-being our own happiness is wholly dependent, and then for the many, unknown to us, to whose destinies we are bound by the ties of sympathy. A hundred times every day I remind myself that my inner and outer life are based on the labors of other men, living and dead, and that I must exert myself in order to give in the same measure as I have received and am still receiving. . . .

Albert Einstein (1931)