Electromagnetic Theory

Electromagnetic Theory notes by K. Sreeman Reddy.

Mathematical Preliminaries

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det\begin{bmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\end{bmatrix}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{A}$$

$$\mathbf{a}\cdot[\mathbf{b}\times\mathbf{c}] = \varepsilon_{ijk}a^i b^j c^k$$

$$[\mathbf{b}\times\mathbf{c}]^i = [\mathbf{b}\times\mathbf{c}]_i = \varepsilon_{ijk}b^j c^k$$

since $a_i = \delta_{ij}a^j = a^i$ for the Cartesian metric $\delta_{ij} = \mathrm{diag}(1,1,1)$. For a fixed $|d\mathbf{r}|$ the direction of gradient will give the maximum increase.

$$df(x) = \big(\nabla f(x)\big)\cdot d\mathbf{r}$$

$$\big(\nabla f(x)\big)\cdot\mathbf{v} = D_{\mathbf{v}}f(x)$$

$$\nabla\cdot\mathbf{F}|_{\mathbf{x_0}} = \lim_{V\to 0}\frac{1}{|V|}\oiint_{S(V)}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

$$(\nabla\times\mathbf{F})(p)\cdot\hat{\mathbf{n}} \overset{\text{def}}{=} \lim_{A\to 0}\frac{1}{|A|}\oint_C \mathbf{F}\cdot d\mathbf{r}$$

Curvilinear coordinates

$$\mathbf{h}_1 = \frac{\partial\mathbf{r}}{\partial q^1};\ \mathbf{h}_2 = \frac{\partial\mathbf{r}}{\partial q^2};\ \mathbf{h}_3 = \frac{\partial\mathbf{r}}{\partial q^3}$$

$$h_1 = |\mathbf{h}_1|;\ h_2 = |\mathbf{h}_2|;\ h_3 = |\mathbf{h}_3|$$

$$\mathbf{v} = v^i\mathbf{h}_i = v_i\mathbf{h}^i$$

Note: In slides prof **only** uses contravariant components (i.e only v^i, $\mathbf{h}_i$, $\hat{\mathbf{e}}_i$ and h_i are used) but v^i is written as v_i.

$$\mathbf{h}_i = \frac{\partial\mathbf{r}}{\partial q^i} \quad \mathbf{h}^i = \nabla q^i \quad \mathbf{h}^i\cdot\mathbf{h}_j = \delta^i_j$$

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = \mathbf{h}_i\cdot\mathbf{h}_j dr^i dr^j = g$$

$$g_{ij} = \mathbf{h}_i\cdot\mathbf{h}_j = g_{ji}\ ;\ \ g^{ij} = \mathbf{h}^i\cdot\mathbf{h}^j = g^{ji}$$

$$\mathbf{u}\cdot\mathbf{v} = u^i v_i = u_i v^i = g_{ij} u^i v^j = g^{ij} u_i v_j$$

Orthogonal coordinate system

$$g_{ij} = 0 \quad \text{if} \quad i \neq j$$

$$\mathbf{h}_i \cdot \mathbf{h}_j = 0 \quad \text{if} \quad i \neq j$$

$$\mathbf{h}^i = \frac{\mathbf{h}_i}{h_i^2} \Rightarrow h^i = \frac{1}{h_i}$$

$$h_k(\mathbf{r}) \stackrel{\text{def}}{=} \sqrt{g_{kk}(\mathbf{r})}$$

- dot product is easier in orthogonal

$$\hat{\mathbf{e}}_i = \hat{\mathbf{e}}^i = \frac{\mathbf{h}_i}{h_i} = h_i \mathbf{h}^i$$

$$\nabla\varphi = \frac{1}{h_i}\frac{\partial\varphi}{\partial q^i}\mathbf{b}^i$$

$$x_i = h_i^2 x^i$$

$$\mathbf{x}\cdot\mathbf{y} = \sum_i h_i^2 x^i y^i = \sum_i \frac{x_i y_i}{h_i^2} = \sum_i x^i y_i = \sum_i x_i y^i$$

- cross product

$$\mathbf{x}\times\mathbf{y} = \sum_i x^i h_i \hat{\mathbf{e}}_i \times \sum_j y^j h_j \hat{\mathbf{e}}_j$$

$$\mathbf{x}\times\mathbf{y} = \left(x^2 y^3 - x^3 y^2\right)\frac{h_2 h_3}{h_1}\mathbf{e}_1 + \left(x^3 y^1 - x^1 y^3\right)\frac{h_1 h_3}{h_2}\mathbf{e}_2 + \left(x^1 y^2 - x^2 y^1\right)\frac{h_1 h_2}{h_3}\mathbf{e}_3$$

Elements

$$d\boldsymbol{\ell} = h_i dq^i \hat{\mathbf{e}}_i = \frac{\partial\mathbf{r}}{\partial q^i} dq^i$$

$$\begin{aligned} d\mathbf{S}_k &= (h_i dq^i \hat{\mathbf{e}}_i) \times (h_j dq^j \hat{\mathbf{e}}_j) \\ &= h_i h_j dq^i dq^j \hat{\mathbf{e}}_k (\text{no summation and i,j,k are different}) \end{aligned}$$

$$\begin{aligned} dV &= |(h_1\, dq^1 \hat{\mathbf{e}}_1) \cdot (h_2\, dq^2 \hat{\mathbf{e}}_2) \times (h_3\, dq^3 \hat{\mathbf{e}}_3)| \\ &= h_1 h_2 h_3\, dq^1\, dq^2\, dq^3 \end{aligned}$$

Gradient

$$\nabla = \mathbf{h}^i \frac{\partial}{\partial q^i} = \frac{\hat{\mathbf{e}}_k}{h_i}\frac{\partial}{\partial q^i}$$

$$\nabla\phi = \sum_k \frac{\hat{\mathbf{e}}_k}{h_k}\frac{\partial\phi}{\partial q^k}$$

$$\nabla\phi = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial\phi}{\partial q^1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial\phi}{\partial q^2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial\phi}{\partial q^3}$$

Divergence

$$\nabla \cdot \mathbf{F} = \frac{1}{\prod_j h_j} \frac{\partial}{\partial q^k} \left(\frac{\prod_j h_j}{h_k} F_k \right)$$

$$\Rightarrow \nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q^1} (F_1 h_2 h_3) + \frac{\partial}{\partial q^2} (F_2 h_3 h_1) + \frac{\partial}{\partial q^3} (F_3 h_1 h_2) \right]$$

Curl

$$\nabla \times \mathbf{F} = \frac{\hat{\mathbf{e}}_k}{\prod_j h_j} \epsilon_{ijk} h_k \frac{\partial}{\partial q^i} (h_j F_j)$$

$$\Rightarrow \nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial q^1} & \dfrac{\partial}{\partial q^2} & \dfrac{\partial}{\partial q^3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} = \frac{\hat{\mathbf{e}}_1}{h_2 h_3} \left[\frac{\partial}{\partial q^2} (h_3 F_3) - \frac{\partial}{\partial q^3} (h_2 F_2) \right]$$
$$+ \frac{\hat{\mathbf{e}}_2}{h_3 h_1} \left[\frac{\partial}{\partial q^3} (h_1 F_1) - \frac{\partial}{\partial q^1} (h_3 F_3) \right] + \frac{\hat{\mathbf{e}}_3}{h_1 h_2} \left[\frac{\partial}{\partial q^1} (h_2 F_2) - \frac{\partial}{\partial q^2} (h_1 F_1) \right]$$

Laplacian

$$\nabla^2 \phi = \frac{1}{\prod_j h_j} \frac{\partial}{\partial q^k} \left(\frac{\prod_j h_j}{h_k^2} \frac{\partial \phi}{\partial q^k} \right)$$

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q^1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial q^1} \right) + \frac{\partial}{\partial q^2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial q^2} \right) + \frac{\partial}{\partial q^3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial q^3} \right) \right]$$

Generating

A simple method for generating orthogonal coordinates systems in two dimensions is by a conformal mapping of a standard two-dimensional grid of Cartesian coordinates (x, y). A complex number $z = x + iy$ can be formed from the real coordinates x and y, where i represents the imaginary unit. Any holomorphic function $w = f(z)$ with non-zero complex derivative will produce a conformal mapping; if the resulting complex number is written $w = u + iv$, then the curves of constant u and v intersect at right angles, just as the original lines of constant x and y did.

Orthogonal coordinates in three and higher dimensions can be generated from an orthogonal two-dimensional coordinate system, either by projecting it into a new dimension (cylindrical coordinates) or by rotating the two-dimensional system about one of its symmetry axes. However, there are other orthogonal coordinate systems in three dimensions that cannot be obtained by projecting or rotating a two-dimensional system, such as the ellipsoidal coordinates. More general orthogonal coordinates may be obtained by starting with some necessary coordinate surfaces and considering their orthogonal trajectories.

Examples:

Table of orthogonal coordinates

Curvillinear coordinates (q_1, q_2, q_3)	Transformation from cartesian (x, y, z)	Scale factors
Spherical polar coordinates $(r, \theta, \phi) \in [0, \infty) \times [0, \pi] \times [0, 2\pi)$	$x = r \sin\theta \cos\phi$ $y = r \sin\theta \sin\phi$ $z = r \cos\theta$	$h_1 = 1$ $h_2 = r$ $h_3 = r \sin\theta$
Cylindrical polar coordinates $(r, \phi, z) \in [0, \infty) \times [0, 2\pi) \times (-\infty, \infty)$	$x = r \cos\phi$ $y = r \sin\phi$ $z = z$	$h_1 = h_3 = 1$ $h_2 = r$

Parabolic cylindrical coordinates $(u,v,z)\in(-\infty,\infty)\times[0,\infty)\times(-\infty,\infty)$	$x=\frac{1}{2}(u^2-v^2)$ $y=uv$ $z=z$	$h_1=h_2=\sqrt{u^2+v^2}$ $h_3=1$
Parabolic coordinates $(u,v,\phi)\in[0,\infty)\times[0,\infty)\times[0,2\pi)$	$x=uv\cos\phi$ $y=uv\sin\phi$ $z=\frac{1}{2}(u^2-v^2)$	$h_1=h_2=\sqrt{u^2+v^2}$ $h_3=uv$
Paraboloidal coordinates $(\lambda,\mu,\nu)\in[0,b^2)\times(b^2,a^2)\times(a^2,\infty)$ $b^2<a^2$	$\frac{x^2}{q_i-a^2}+\frac{y^2}{q_i-b^2}=2z+q_i$ where $(q_1,q_2,q_3)=(\lambda,\mu,\nu)$	$h_i=\frac{1}{2}\sqrt{\frac{(q_j-q_i)(q_k-q_i)}{(a^2-q_i)(b^2-q_i)}}$
Ellipsoidal coordinates $(\lambda,\mu,\nu)\in[0,c^2)\times(c^2,b^2)\times(b^2,a^2)$ $\lambda<c^2<b^2<a^2,$ $c^2<\mu<b^2<a^2,$ $c^2<b^2<\nu<a^2,$	$\frac{x^2}{a^2-q_i}+\frac{y^2}{b^2-q_i}+\frac{z^2}{c^2-q_i}=1$ where $(q_1,q_2,q_3)=(\lambda,\mu,\nu)$	$h_i=\frac{1}{2}\sqrt{\frac{(q_j-q_i)(q_k-q_i)}{(a^2-q_i)(b^2-q_i)(c^2-q_i)}}$
Elliptic cylindrical coordinates $(u,v,z)\in[0,\infty)\times[0,2\pi)\times(-\infty,\infty)$	$x=a\cosh u\cos v$ $y=a\sinh u\sin v$ $z=z$	$h_1=h_2=a\sqrt{\sinh^2 u+\sin^2 v}$ $h_3=1$
Prolate spheroidal coordinates $(\xi,\eta,\phi)\in[0,\infty)\times[0,\pi]\times[0,2\pi)$	$x=a\sinh\xi\sin\eta\cos\phi$ $y=a\sinh\xi\sin\eta\sin\phi$ $z=a\cosh\xi\cos\eta$	$h_1=h_2=a\sqrt{\sinh^2\xi+\sin^2\eta}$ $h_3=a\sinh\xi\sin\eta$
Oblate spheroidal coordinates $(\xi,\eta,\phi)\in[0,\infty)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi)$	$x=a\cosh\xi\cos\eta\cos\phi$ $y=a\cosh\xi\cos\eta\sin\phi$ $z=a\sinh\xi\sin\eta$	$h_1=h_2=a\sqrt{\sinh^2\xi+\sin^2\eta}$ $h_3=a\cosh\xi\cos\eta$
Bipolar cylindrical coordinates $(u,v,z)\in[0,2\pi)\times(-\infty,\infty)\times(-\infty,\infty)$	$x=\frac{a\sinh v}{\cosh v-\cos u}$ $y=\frac{a\sin u}{\cosh v-\cos u}$ $z=z$	$h_1=h_2=\frac{a}{\cosh v-\cos u}$ $h_3=1$
Toroidal coordinates $(u,v,\phi)\in(-\pi,\pi]\times[0,\infty)\times[0,2\pi)$	$x=\frac{a\sinh v\cos\phi}{\cosh v-\cos u}$ $y=\frac{a\sinh v\sin\phi}{\cosh v-\cos u}$ $z=\frac{a\sin u}{\cosh v-\cos u}$	$h_1=h_2=\frac{a}{\cosh v-\cos u}$ $h_3=\frac{a\sinh v}{\cosh v-\cos u}$
Bispherical coordinates $(u,v,\phi)\in(-\pi,\pi]\times[0,\infty)\times[0,2\pi)$	$x=\frac{a\sin u\cos\phi}{\cosh v-\cos u}$ $y=\frac{a\sin u\sin\phi}{\cosh v-\cos u}$ $z=\frac{a\sinh v}{\cosh v-\cos u}$	$h_1=h_2=\frac{a}{\cosh v-\cos u}$ $h_3=\frac{a\sin u}{\cosh v-\cos u}$
Conical coordinates (λ,μ,ν) $\nu^2<b^2<\mu^2<a^2$ $\lambda\in[0,\infty)$	$x=\frac{\lambda\mu\nu}{ab}$ $y=\frac{\lambda}{a}\sqrt{\frac{(\mu^2-a^2)(\nu^2-a^2)}{a^2-b^2}}$ $z=\frac{\lambda}{b}\sqrt{\frac{(\mu^2-b^2)(\nu^2-b^2)}{b^2-a^2}}$	$h_1=1$ $h_2^2=\frac{\lambda^2(\mu^2-\nu^2)}{(\mu^2-a^2)(b^2-\mu^2)}$ $h_3^2=\frac{\lambda^2(\mu^2-\nu^2)}{(\nu^2-a^2)(\nu^2-b^2)}$

Del in cylindrical and spherical coordinates

Conversion between Cartesian, cylindrical, and spherical coordinates[1]

		From		
		Cartesian	Cylindrical	Spherical
To	Cartesian	N/A	$x = \rho\cos\varphi$ $y = \rho\sin\varphi$ $z = z$	$x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$
	Cylindrical	$\rho = \sqrt{x^2 + y^2}$ $\varphi = \arctan\left(\frac{y}{x}\right)$ $z = z$	N/A	$\rho = r\sin\theta$ $\varphi = \varphi$ $z = r\cos\theta$
	Spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$ $\varphi = \arctan\left(\frac{y}{x}\right)$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan\left(\frac{\rho}{z}\right)$ $\varphi = \varphi$	N/A

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of *destination* coordinates[1]

	Cartesian	Cylindrical	Spherical
Cartesian	N/A	$\hat{\mathbf{x}} = \cos\varphi\hat{\boldsymbol{\rho}} - \sin\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\varphi\hat{\boldsymbol{\rho}} + \cos\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
Cylindrical	$\hat{\boldsymbol{\rho}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{\boldsymbol{\varphi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\boldsymbol{\rho}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
Spherical	$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}\right)z - \left(x^2 + y^2\right)\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}\sqrt{x^2 + y^2}}$ $\hat{\boldsymbol{\varphi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$	$\hat{\mathbf{r}} = \frac{\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{z\hat{\boldsymbol{\rho}} - \rho\hat{\mathbf{z}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of *source* coordinates

	Cartesian	Cylindrical	Spherical
Cartesian	N/A	$\hat{\mathbf{x}} = \dfrac{x\hat{\boldsymbol{\rho}} - y\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}}$ $\hat{\mathbf{y}} = \dfrac{y\hat{\boldsymbol{\rho}} + x\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \dfrac{x\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) - y\sqrt{x^2+y^2+z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$ $\hat{\mathbf{y}} = \dfrac{y\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) + x\sqrt{x^2+y^2+z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$ $\hat{\mathbf{z}} = \dfrac{z\hat{\mathbf{r}} - \sqrt{x^2+y^2}\hat{\boldsymbol{\theta}}}{\sqrt{x^2+y^2+z^2}}$
Cylindrical	$\hat{\boldsymbol{\rho}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}$ $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\boldsymbol{\rho}} = \dfrac{\rho\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}}{\sqrt{\rho^2+z^2}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \dfrac{z\hat{\mathbf{r}} - \rho\hat{\boldsymbol{\theta}}}{\sqrt{\rho^2+z^2}}$
Spherical	$\hat{\mathbf{r}} = \sin\theta\left(\cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}\right) + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos\theta\left(\cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}\right) - \sin\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$	$\hat{\mathbf{r}} = \sin\theta\hat{\boldsymbol{\rho}} + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos\theta\hat{\boldsymbol{\rho}} - \sin\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Table with the del operator in cartesian, cylindrical and spherical coordinates

Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates (ρ, φ, z)	Spherical coordinates (r, θ, φ), where θ is the polar angle and φ is the azimuthal angle[α]
Vector field $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$	$A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$
Gradient ∇f[1]	$\dfrac{\partial f}{\partial x}\hat{\mathbf{x}} + \dfrac{\partial f}{\partial y}\hat{\mathbf{y}} + \dfrac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\dfrac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \dfrac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\dfrac{\partial f}{\partial r}\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
Divergence $\nabla\cdot\mathbf{A}$[1]	$\dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{\rho}\dfrac{\partial(\rho A_\rho)}{\partial \rho} + \dfrac{1}{\rho}\dfrac{\partial A_\varphi}{\partial \varphi} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{r^2}\dfrac{\partial(r^2 A_r)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(A_\theta\sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times\mathbf{A}$[1]	$\left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}}$ $+\left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}}$ $+\left(\dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\dfrac{1}{\rho}\dfrac{\partial A_z}{\partial\varphi} - \dfrac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}}$ $+\left(\dfrac{\partial A_\rho}{\partial z} - \dfrac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\varphi}}$ $+\dfrac{1}{\rho}\left(\dfrac{\partial(\rho A_\varphi)}{\partial\rho} - \dfrac{\partial A_\rho}{\partial\varphi}\right)\hat{\mathbf{z}}$	$\dfrac{1}{r\sin\theta}\left(\dfrac{\partial}{\partial\theta}(A_\varphi\sin\theta) - \dfrac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}}$ $+\dfrac{1}{r}\left(\dfrac{1}{\sin\theta}\dfrac{\partial A_r}{\partial\varphi} - \dfrac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}}$ $+\dfrac{1}{r}\left(\dfrac{\partial}{\partial r}(rA_\theta) - \dfrac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$
Laplace operator $\nabla^2 f \equiv \Delta f$[1]	$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$	$\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial f}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial\varphi^2} + \dfrac{\partial^2 f}{\partial z^2}$	$\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial f}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 f}{\partial\varphi^2}$
Vector Laplacian $\nabla^2\mathbf{A} \equiv \Delta\mathbf{A}$[2]	$\nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$	$\left(\nabla^2 A_\rho - \dfrac{A_\rho}{\rho^2} - \dfrac{2}{\rho^2}\dfrac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\rho}}$ $+\left(\nabla^2 A_\varphi - \dfrac{A_\varphi}{\rho^2} + \dfrac{2}{\rho^2}\dfrac{\partial A_\rho}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}}$ $+\nabla^2 A_z\hat{\mathbf{z}}$	$\left(\nabla^2 A_r - \dfrac{2A_r}{r^2} - \dfrac{2}{r^2\sin\theta}\dfrac{\partial(A_\theta\sin\theta)}{\partial\theta} - \dfrac{2}{r^2\sin\theta}\dfrac{\partial A_\varphi}{\partial\varphi}\right)\hat{\mathbf{r}}$ $+\left(\nabla^2 A_\theta - \dfrac{A_\theta}{r^2\sin^2\theta} + \dfrac{2}{r^2}\dfrac{\partial A_r}{\partial\theta} - \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\theta}}$ $+\left(\nabla^2 A_\varphi - \dfrac{A_\varphi}{r^2\sin^2\theta} + \dfrac{2}{r^2\sin\theta}\dfrac{\partial A_r}{\partial\varphi} + \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial A_\theta}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}}$
Material derivative[α][3] $(\mathbf{A}\cdot\nabla)\mathbf{B}$	$\mathbf{A}\cdot\nabla B_x\hat{\mathbf{x}} + \mathbf{A}\cdot\nabla B_y\hat{\mathbf{y}} + \mathbf{A}\cdot\nabla B_z\hat{\mathbf{z}}$	$\left(A_\rho\dfrac{\partial B_\rho}{\partial\rho} + \dfrac{A_\varphi}{\rho}\dfrac{\partial B_\rho}{\partial\varphi} + A_z\dfrac{\partial B_\rho}{\partial z} - \dfrac{A_\varphi B_\varphi}{\rho}\right)\hat{\boldsymbol{\rho}}$ $+\left(A_\rho\dfrac{\partial B_\varphi}{\partial\rho} + \dfrac{A_\varphi}{\rho}\dfrac{\partial B_\varphi}{\partial\varphi} + A_z\dfrac{\partial B_\varphi}{\partial z} + \dfrac{A_\varphi B_\rho}{\rho}\right)\hat{\boldsymbol{\varphi}}$ $+\left(A_\rho\dfrac{\partial B_z}{\partial\rho} + \dfrac{A_\varphi}{\rho}\dfrac{\partial B_z}{\partial\varphi} + A_z\dfrac{\partial B_z}{\partial z}\right)\hat{\mathbf{z}}$	$\left(A_r\dfrac{\partial B_r}{\partial r} + \dfrac{A_\theta}{r}\dfrac{\partial B_r}{\partial\theta} + \dfrac{A_\varphi}{r\sin\theta}\dfrac{\partial B_r}{\partial\varphi} - \dfrac{A_\theta B_\theta + A_\varphi B_\varphi}{r}\right)\hat{\mathbf{r}}$ $+\left(A_r\dfrac{\partial B_\theta}{\partial r} + \dfrac{A_\theta}{r}\dfrac{\partial B_\theta}{\partial\theta} + \dfrac{A_\varphi}{r\sin\theta}\dfrac{\partial B_\theta}{\partial\varphi} + \dfrac{A_\theta B_r}{r} - \dfrac{A_\varphi B_\varphi\cot\theta}{r}\right)\hat{\boldsymbol{\theta}}$ $+\left(A_r\dfrac{\partial B_\varphi}{\partial r} + \dfrac{A_\theta}{r}\dfrac{\partial B_\varphi}{\partial\theta} + \dfrac{A_\varphi}{r\sin\theta}\dfrac{\partial B_\varphi}{\partial\varphi} + \dfrac{A_\varphi B_r}{r} + \dfrac{A_\varphi B_\theta\cot\theta}{r}\right)\hat{\boldsymbol{\varphi}}$
Tensor $\nabla\cdot\mathbf{T}$ (not to be confused with 2nd order tensor divergence)	$\left(\dfrac{\partial T_{xx}}{\partial x} + \dfrac{\partial T_{yx}}{\partial y} + \dfrac{\partial T_{zx}}{\partial z}\right)\hat{\mathbf{x}}$ $+\left(\dfrac{\partial T_{xy}}{\partial x} + \dfrac{\partial T_{yy}}{\partial y} + \dfrac{\partial T_{zy}}{\partial z}\right)\hat{\mathbf{y}}$ $+\left(\dfrac{\partial T_{xz}}{\partial x} + \dfrac{\partial T_{yz}}{\partial y} + \dfrac{\partial T_{zz}}{\partial z}\right)\hat{\mathbf{z}}$	$\left[\dfrac{\partial T_{\rho\rho}}{\partial\rho} + \dfrac{1}{\rho}\dfrac{\partial T_{\varphi\rho}}{\partial\varphi} + \dfrac{\partial T_{z\rho}}{\partial z} + \dfrac{1}{\rho}(T_{\rho\rho} - T_{\varphi\varphi})\right]\hat{\boldsymbol{\rho}}$ $+\left[\dfrac{\partial T_{\rho\varphi}}{\partial\rho} + \dfrac{1}{\rho}\dfrac{\partial T_{\varphi\varphi}}{\partial\varphi} + \dfrac{\partial T_{z\varphi}}{\partial z} + \dfrac{1}{\rho}(T_{\rho\varphi} + T_{\varphi\rho})\right]\hat{\boldsymbol{\varphi}}$ $+\left[\dfrac{\partial T_{\rho z}}{\partial\rho} + \dfrac{1}{\rho}\dfrac{\partial T_{\varphi z}}{\partial\varphi} + \dfrac{\partial T_{zz}}{\partial z} + \dfrac{T_{\rho z}}{\rho}\right]\hat{\mathbf{z}}$	$\left[\dfrac{\partial T_{rr}}{\partial r} + 2\dfrac{T_{rr}}{r} + \dfrac{1}{r}\dfrac{\partial T_{\theta r}}{\partial\theta} + \dfrac{\cot\theta}{r}T_{\theta r} + \dfrac{1}{r\sin\theta}\dfrac{\partial T_{\varphi r}}{\partial\varphi} - \dfrac{1}{r}(T_{\theta\theta} + T_{\varphi\varphi})\right]\hat{\mathbf{r}}$ $+\left[\dfrac{\partial T_{r\theta}}{\partial r} + 2\dfrac{T_{r\theta}}{r} + \dfrac{1}{r}\dfrac{\partial T_{\theta\theta}}{\partial\theta} + \dfrac{\cot\theta}{r}T_{\theta\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial T_{\varphi\theta}}{\partial\varphi} + \dfrac{T_{\theta r}}{r} - \dfrac{\cot\theta}{r}T_{\varphi\varphi}\right]\hat{\boldsymbol{\theta}}$ $+\left[\dfrac{\partial T_{r\varphi}}{\partial r} + 2\dfrac{T_{r\varphi}}{r} + \dfrac{1}{r}\dfrac{\partial T_{\theta\varphi}}{\partial\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial T_{\varphi\varphi}}{\partial\varphi} + \dfrac{T_{\varphi r}}{r} + \dfrac{\cot\theta}{r}(T_{\theta\varphi} + T_{\varphi\theta})\right]\hat{\boldsymbol{\varphi}}$
Differential displacement $d\boldsymbol{\ell}$[1]	$dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$	$d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\varphi\,\hat{\boldsymbol{\varphi}} + dz\,\hat{\mathbf{z}}$	$dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,d\varphi\,\hat{\boldsymbol{\varphi}}$
Differential normal area $d\mathbf{S}$	$dy\,dz\,\hat{\mathbf{x}}$ $+\,dx\,dz\,\hat{\mathbf{y}}$ $+\,dx\,dy\,\hat{\mathbf{z}}$	$\rho\,d\varphi\,dz\,\hat{\boldsymbol{\rho}}$ $+\,d\rho\,dz\,\hat{\boldsymbol{\varphi}}$ $+\,\rho\,d\rho\,d\varphi\,\hat{\mathbf{z}}$	$r^2\sin\theta\,d\theta\,d\varphi\,\hat{\mathbf{r}}$ $+\,r\sin\theta\,dr\,d\varphi\,\hat{\boldsymbol{\theta}}$ $+\,r\,dr\,d\theta\,\hat{\boldsymbol{\varphi}}$
Differential volume dV[1]	$dx\,dy\,dz$	$\rho\,d\rho\,d\varphi\,dz$	$r^2\sin\theta\,dr\,d\theta\,d\varphi$

Note: In $2D$ polar formulas are similar to $3D$ cylindrical not $3D$ polar.

Divergence theorem

Suppose V is a subset of $\mathbb{R}^n$ which is compact and has a piecewise smooth boundary S or ∂V. If F is a continuously differentiable vector field defined on a neighbourhood of V, then

$$\iiint_V (\nabla \cdot \mathbf{F})\, \mathrm{d}V = (\mathbf{F} \cdot \hat{\mathbf{n}})\, \mathrm{d}S$$

$$\iiint_V \text{total of the sources in the volume } = \text{total flow across the boundary}\, \mathrm{d}S$$

Stokes' theorem

$$\iint_\Sigma (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} = \oint_{\partial\Sigma} \mathbf{A} \cdot \mathrm{d}\mathbf{l}.$$

- the RHS is invariant under the change of surface as long as the boundary is same

Dirac delta

Wrongly if you apply divergence theorem for the below equation you get that the divergence is 0.

$$\boxed{\begin{aligned} \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) &= 4\pi\delta^3(r) \\ \nabla \left(\frac{1}{r}\right) &= -\frac{\hat{r}}{r^2} \\ \nabla^2 \left(\frac{1}{r}\right) &= -4\pi\delta^3(r) \end{aligned}}$$

$$\boxed{\nabla^2 \left(\frac{1}{\|\mathbf{r} - \mathbf{r}_0\|}\right) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)} \tag{1}$$

$$\begin{aligned} \delta'(x) &= -\delta'(-x) \\ x\delta'(x) &= -\delta(x) \\ x^2\delta'(x) &= 0 \\ x^2\delta''(x) &= 2\delta(x) \end{aligned}$$

Introduction

Name	Integral equations	Differential equations
Gauss's law	$\mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \iiint_\Omega \rho\, \mathrm{d}V$	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
Gauss's law for magnetism	$\mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$	$\nabla \cdot \mathbf{B} = 0$
Faraday's law of induction	$\oint_{\partial\Sigma} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_\Sigma \mathbf{B} \cdot \mathrm{d}\mathbf{S}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

Name	[Integral](#) equations	[Differential](#) equations
Ampère's circuital law	$\oint_{\partial\Sigma} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \left(\iint_{\Sigma} \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Sigma} \mathbf{E} \cdot \mathrm{d}\mathbf{S} \right)$	$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$

Note: In $2D$ electromagnetism is very different from $3D$. In $2D$ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and there are 3 independent components. 2 of them will be time-space and make a $2D$ electric field vector. The remaining component is space-space and is pseudo-scalar magnetic field. Remember that cross product will give pseudo-vector only in $3D$. In n dimensions to get a unique vector perpendicular to the given vectors we need $n-1$ vectors.

- In this course when the say $2D$ electromagnetism that means neglect the z direction but still in $3D$.

Laplace's Equation

$$\nabla^2 V = 0 \qquad \text{or} \qquad \Delta V = 0,$$

- for a general charge distribution V can be calculated more easily that $\mathbf{E}$.
- even when the charge distribution is not known it is easier to work with potentials by confining our attention to places where there is no charge.
- electrostatics is the study of Laplace's equation
- the more general version $\nabla^2 \varphi = -\frac{\rho}{\varepsilon}$ is called **Poisson's equation**.
- we cannot write down a "general closed form solution" for Laplace's Equation in more than $1D$

The mean value theorem

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{spherical surface}} V dS$$

- immediately it follows that the average value in the entire spherical volume is also same since each layer of spherical surface has same value.
- Proof: Using divergence theorem we can show that the surface integral will be independent of R

$$\int_{\text{vol}} \vec{\nabla} \cdot (\vec{\nabla} V) d\tau = \int_{\text{surface}} \vec{\nabla} V \cdot d\vec{S}$$
$$0 = R^2 \left(\frac{\partial}{\partial r} \int_{\text{surface}} V(r, \theta, \phi) \sin\theta d\theta d\phi \right) \Bigg|_R$$

τ for volume because V is used for potential.

Earnshaw's theorem states that a collection of point charges cannot be maintained in a stable stationary equilibrium configuration solely by the electrostatic interaction of the charges.

- saddle points are possible
- maxima and minima are not possible

First uniqueness theorem: The solution to Laplace's equation in some volume V is uniquely determined if V is specified on the boundary surface S.

- **Dirichlet boundary conditions** specify the value of the potential at each surface point.

Second uniqueness theorem: In a volume V surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given. The region as a whole can be bounded by another conductor, or else unbounded.

- **Neumann boundary conditions** specify the value of the normal component of the gradient of the potential at each surface point.

Mixed boundary conditions: At some points V and at other points $\hat{\mathbf{n}} \cdot \nabla V$ is given. Here also unique.

- giving both V and $\hat{\mathbf{n}} \cdot \nabla V$ makes it over determined.

Poisson 2D formula

For $2D$ if we give the values of V on a circle then the **entire potential function** will be fixed.

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

Try $V = R(r)e^{im\theta}$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - m^2R = 0$$

For $m \neq 0$ $R = Ar^{\pm m}$ and for $m = 0$ $R = A_0 + B_0 \ln r$. The general solution will be

$$V(r,\theta) = (A_0 + B_0 \ln r) + \sum_{m=-\infty,\neq 0}^{\infty}\left(A_m r^{|m|} + \frac{B_m}{r^{|m|}}\right)e^{im\theta}$$

Of course for **inside the circle** neglect the $\frac{B_m}{r^{|m|}}$ and $B_0 \ln r$ terms and for **outside the circle** neglect the $A_m r^{|m|}$ and $B_0 \ln r$ terms. For $r < 1$ we get

$$V(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\alpha f(\alpha)\left(\frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}\right)$$

Polar coordinates

In $2D$ **the most general solution** in polar coordinates is

$$\varphi(\rho,\phi) = (A_0 + B_0 \ln \rho)(C_0 + D_0\phi) + \sum_{\alpha=1}^{\infty}\left[A_\alpha \rho^\alpha + B_\alpha \rho^{-\alpha}\right]\left[C_\alpha \sin \alpha\phi + D_\alpha \cos \alpha\phi\right].$$

Cylindrical coordinates

Conformal mapping

- applies only to two-dimensional potentials. These are systems in which V depends only on x and y, for example, all conducting boundaries being cylinders with elements running parallel to z.

Significance of the cylindrical co-ordinate
Off-axis expansion (electrostatic lensing)
Bessel functions

Green's function

$$\mathrm{L}\ G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$$

we generally take $L=\nabla^2$.

- Green's function is not unique.

1D and 2D

$$\begin{aligned}
G_{1\mathrm{D}} &= -\frac{1}{2}|\mathbf{r}-\mathbf{r}'| = -\frac{1}{2}|x-x'| \\
G_{2\mathrm{D}} &= -\frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'| \\
G_{3\mathrm{D}} &= \frac{1}{4\pi}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)
\end{aligned}$$

check the **dimensions**. For $2D$ dimensions cannot be correct unless we add the constant $\frac{1}{2\pi}\ln d$. Here we **can't uniquely** decide the constant because for $2D$ case at ∞ the G **can never** be 0.

$F_{\mu\nu}$ is antisymmetric. In $1D+1$ it only has E. In $2D+1$ it has E_x, E_y, B. B being a pseudo scalar. In $3D+1$ it has $E_x, E_y, E_z, B_x, B_y, B_z$ with $\vec{B}$ being a pseudo vector. In $2D$ Lorentz force ($\dfrac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = qF^{\alpha\beta}U_\beta$) becomes $\mathbf{F} = q\,\mathbf{E} + Bq\,\mathbf{v}\times\hat{\mathbf{k}}$ where $\hat{\mathbf{k}}$ is an imaginary direction.

Conservation laws

$$\frac{\partial u_{\mathrm{em}}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E} = 0$$

$$\frac{\partial \mathbf{p}_{\mathrm{em}}}{\partial t} - \boldsymbol{\nabla}\cdot\sigma + \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} = 0 \ \Leftrightarrow\ \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t} - \nabla\cdot\sigma + \mathbf{f} = 0$$

$$\sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j - \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\delta_{ij}$$

$$T^{\mu\nu} = \begin{bmatrix}
\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) & \frac{1}{c}S_{\mathrm{x}} & \frac{1}{c}S_{\mathrm{y}} & \frac{1}{c}S_{\mathrm{z}} \\
\frac{1}{c}S_{\mathrm{x}} & -\sigma_{\mathrm{xx}} & -\sigma_{\mathrm{xy}} & -\sigma_{\mathrm{xz}} \\
\frac{1}{c}S_{\mathrm{y}} & -\sigma_{\mathrm{yx}} & -\sigma_{\mathrm{yy}} & -\sigma_{\mathrm{yz}} \\
\frac{1}{c}S_{\mathrm{z}} & -\sigma_{\mathrm{zx}} & -\sigma_{\mathrm{zy}} & -\sigma_{\mathrm{zz}}
\end{bmatrix},$$

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \mathbf{E}\times\mathbf{H}$$

The Poynting vector S has dimensions of (energy/volume) × velocity. This invites us to interpret S as an energy current density by analogy with the usual charge current density.

Energy

The flux of electromagnetic energy density is

$$u_{\mathrm{em}} = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$$

$$U_{\text{em}} = \int d^3r \left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right)$$

Momentum

The electromagnetic momentum density is

$$\mathbf{p}_{\text{em}} = \frac{\mathbf{S}}{c^2} = \epsilon_0 \mathbf{E} \times \mathbf{B}$$

$$\mathbf{P}_{\text{em}} = \epsilon_0 \int d^3r \, \mathbf{E} \times \mathbf{B}$$

Angular momentum

$$\mathbf{L}_{\text{em}} = \epsilon_0 \int d^3r \, \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$$

Abraham–Minkowski controversy (in medium)

Moving charges and radiation

$$A^\alpha = \left(\frac{1}{c}\phi, \mathbf{A} \right)$$

Retarded potential

If we take the **Lorenz gauge** condition: $\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$ then

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3\mathbf{r}'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3\mathbf{r}'$$

$$t_r = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}$$

Jefimenko's equations

Using $\mathbf{E} = -\nabla\varphi - \dfrac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$ we get

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \left[\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}', t_r) + \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} \frac{1}{c} \frac{\partial \rho(\mathbf{r}', t_r)}{\partial t} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{1}{c^2} \frac{\partial \mathbf{J}(\mathbf{r}', t_r)}{\partial t} \right] dV'$$

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \times \mathbf{J}(\mathbf{r}', t_r) + \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} \times \frac{1}{c} \frac{\partial \mathbf{J}(\mathbf{r}', t_r)}{\partial t} \right] dV'$$

Point charge

Liénard–Wiechert potential

For a charge with trajectory given by $\mathbf{r}_s(t')$

$$\rho(\mathbf{r}', t') = q\delta^3(\mathbf{r}' - \mathbf{r}_s(t'))$$

$$\mathbf{J}(\mathbf{r}', t') = q\mathbf{v}_s(t')\delta^3(\mathbf{r'} - \mathbf{r}_s(t'))$$

we get

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{(1 - \mathbf{n}_s \cdot \boldsymbol{\beta}_s)|\mathbf{r} - \mathbf{r}_s|}\right)_{t_r}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 c}{4\pi}\left(\frac{q\boldsymbol{\beta}_s}{(1 - \mathbf{n}_s \cdot \boldsymbol{\beta}_s)|\mathbf{r} - \mathbf{r}_s|}\right)_{t_r} = \frac{\boldsymbol{\beta}_s(t_r)}{c}\varphi(\mathbf{r}, t)$$

The symbol $(\cdots)_{t_r}$ means that the quantities inside the parenthesis should be evaluated at the retarded time $t_r = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}_s(t_r)|$.

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q(\mathbf{n}_s - \boldsymbol{\beta}_s)}{\gamma^2(1 - \mathbf{n}_s \cdot \boldsymbol{\beta}_s)^3|\mathbf{r} - \mathbf{r}_s|^2} + \frac{q\mathbf{n}_s \times \big((\mathbf{n}_s - \boldsymbol{\beta}_s) \times \dot{\boldsymbol{\beta}}_s\big)}{c(1 - \mathbf{n}_s \cdot \boldsymbol{\beta}_s)^3|\mathbf{r} - \mathbf{r}_s|}\right)_{t_r}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\left(\frac{qc(\boldsymbol{\beta}_s \times \mathbf{n}_s)}{\gamma^2(1 - \mathbf{n}_s \cdot \boldsymbol{\beta}_s)^3|\mathbf{r} - \mathbf{r}_s|^2} + \frac{q\mathbf{n}_s \times \Big(\mathbf{n}_s \times \big((\mathbf{n}_s - \boldsymbol{\beta}_s) \times \dot{\boldsymbol{\beta}}_s\big)\Big)}{(1 - \mathbf{n}_s \cdot \boldsymbol{\beta}_s)^3|\mathbf{r} - \mathbf{r}_s|}\right)_{t_r} = \frac{\mathbf{n}_s(t_r)}{c} \times \mathbf{E}(\mathbf{r}, t)$$

here $\boldsymbol{\beta}_s(t) = \frac{\mathbf{v}_s(t)}{c}$, $\mathbf{n}_s(t) = \frac{\mathbf{r} - \mathbf{r}_s(t)}{|\mathbf{r} - \mathbf{r}_s(t)|}$ and $\gamma(t) = \frac{1}{\sqrt{1 - |\boldsymbol{\beta}_s(t)|^2}}$.

Larmor formula

$$P = \frac{2q^2\gamma^6}{3c}\left[(\dot{\boldsymbol{\beta}})^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2\right].$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\frac{|\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^5} = \frac{q^2 a^2}{4\pi c^3}\frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}$$

$$\boxed{P = \frac{2}{3}\frac{q^2}{4\pi\varepsilon_0 c}\left(\frac{\dot{v}}{c}\right)^2 = \frac{2}{3}\frac{q^2 a^2}{4\pi\varepsilon_0 c^3} = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} \text{ (SI units)}}$$

Radiation reaction (Abraham–Lorentz force)

- the electromagnetic force which a radiating system exerts on itself

$$\mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}} = \frac{q^2}{6\pi\varepsilon_0 c^3}\dot{\mathbf{a}} = \frac{2}{3}\frac{q^2}{4\pi\varepsilon_0 c^3}\dot{\mathbf{a}}$$

Appendix

Maxwell's equations

Name	Integral equations	Differential equations
Gauss's law	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\iiint_\Omega \rho \,\mathrm{d}V$	$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$
Gauss's law for magnetism	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0$	$\nabla\cdot\mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_\Sigma \mathbf{B}\cdot\mathrm{d}\mathbf{S}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \mu_0\left(\iint_\Sigma \mathbf{J}\cdot\mathrm{d}\mathbf{S} + \varepsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\iint_\Sigma \mathbf{E}\cdot\mathrm{d}\mathbf{S}\right)$	$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)$

Name	Integral equations (SI convention)	Differential equations (SI convention)	Differential equations (Gaussian convention)
Gauss's law	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{D}\cdot \mathrm{d}\mathbf{S} = \iiint_\Omega \rho_\mathrm{f}\,\mathrm{d}V$	$\nabla\cdot\mathbf{D} = \rho_\mathrm{f}$	$\nabla\cdot\mathbf{D} = 4\pi\rho_\mathrm{f}$
Gauss's law for magnetism	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\nabla\cdot\mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{d}{dt}\iint_\Sigma \mathbf{B}\cdot\mathrm{d}\mathbf{S}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$	$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{H}\cdot \mathrm{d}\boldsymbol{\ell} = \iint_\Sigma \mathbf{J}_\mathrm{f}\cdot\mathrm{d}\mathbf{S} + \frac{d}{dt}\iint_\Sigma \mathbf{D}\cdot\mathrm{d}\mathbf{S}$	$\nabla\times\mathbf{H} = \mathbf{J}_\mathrm{f} + \frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{H} = \frac{1}{c}\left(4\pi\mathbf{J}_\mathrm{f} + \frac{\partial\mathbf{D}}{\partial t}\right)$

Magnetic dipole

$$\mathbf{m} = I\,A\,\hat{\mathbf{n}}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r^2}\frac{\mathbf{m}\times\mathbf{r}}{r} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3},$$

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{r}(\mathbf{m}\cdot\mathbf{r})}{r^5} - \frac{\mathbf{m}}{r^3}\right].$$

$$\boxed{\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})}$$

$$\boxed{\mathbf{N} = \mathbf{m}\times\mathbf{B},}$$

Electric dipole

$$U = -\mathbf{p}\cdot\mathbf{E}, \qquad \boldsymbol{\tau} = \mathbf{p}\times\mathbf{E}$$

$$\mathbf{p}(\mathbf{r}) = \int_V \rho(\mathbf{r}')\,(\mathbf{r}'-\mathbf{r})\,d^3\mathbf{r}',$$

$$\phi(\mathbf{R}) = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{d}\cdot\hat{\mathbf{R}}}{R^2} + \mathcal{O}\left(\frac{d^3}{R^3}\right) \approx \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{R^2},$$

$$\mathbf{E}\left(\mathbf{R}\right)=\frac{3\left(\mathbf{p}\cdot\hat{\mathbf{R}}\right)\hat{\mathbf{R}}-\mathbf{p}}{4\pi\varepsilon_0 R^3}\,.$$

$$\vec{F}=-\nabla U=-\nabla(\vec{p}\cdot\vec{E})=(\vec{p}\cdot\nabla)\vec{E}.$$

Lorentz boost

$$B(\mathbf{v})=\begin{bmatrix}\gamma & -\gamma v_x/c & -\gamma v_y/c & -\gamma v_z/c\\ -\gamma v_x/c & 1+(\gamma-1)\dfrac{v_x^2}{v^2} & (\gamma-1)\dfrac{v_x v_y}{v^2} & (\gamma-1)\dfrac{v_x v_z}{v^2}\\ -\gamma v_y/c & (\gamma-1)\dfrac{v_y v_x}{v^2} & 1+(\gamma-1)\dfrac{v_y^2}{v^2} & (\gamma-1)\dfrac{v_y v_z}{v^2}\\ -\gamma v_z/c & (\gamma-1)\dfrac{v_z v_x}{v^2} & (\gamma-1)\dfrac{v_z v_y}{v^2} & 1+(\gamma-1)\dfrac{v_z^2}{v^2}\end{bmatrix},$$

Holomorphic or complex analytic examples

All polynomial functions in z with complex coefficients are entire functions (holomorphic in the whole complex plane C), and so are the exponential function exp z and the trigonometric functions cos $\cos z=\frac{1}{2}\Big(\exp(iz)+\exp(-iz)\Big)$ and $\sin z=-\frac{1}{2}i\Big(\exp(iz)-\exp(-iz)\Big)$ (cf. Euler's formula). The principal branch of the complex logarithm function log z is holomorphic on the domain C \ {z ∈ R : z ≤ 0}. The square root function can be defined as $\sqrt{z}=\exp\left(\frac{1}{2}\log z\right)$ and is therefore holomorphic wherever the logarithm log z is. The reciprocal function 1/z is holomorphic on C \ {0}. (The reciprocal function, and any other rational function, is meromorphic on C.)

As a consequence of the Cauchy–Riemann equations, any real-valued holomorphic function must be constant. Therefore, the absolute value |z|, the argument arg(z), the real part Re(z) and the imaginary part Im(z) are not holomorphic. Another typical example of a continuous function which is not holomorphic is the complex conjugate $\overline{z}$. (The complex conjugate is antiholomorphic.)