AdS/CFT correspondence and the information paradox

Kasi Reddy Sreeman Reddy
Undergraduate student
Project supervised by Prof P. Ramadevi

IIT Bombay

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Introduction

1. AdS/CFT correspondence is a surprising duality between a string theory and a quantum field theory. QFT and string theories are widely different frameworks.
2. Previously discovered dualities were either between 2 QFTs (Montonen–Olive duality,Seiberg duality etc) or between 2 string theories (S,T,U-dualities)
3. AdS/CFT is a strong–weak duality.
4. Teaches non-perturbative nature of string theory and strongly coupled quantum field theories.

The AdS/CFT correspondence

A big hint related to that the AdS/CFT correspondence is AdS_{d+1} and CFT_d both have the same $SO(2,d)$ symmetry.
In the most popular version of the AdS/CFT correspondence, the gravity side is **asymptotically** $AdS_5 \times S^5$ with $SO(4,2)$ symmetry on the non-compact dimensions and $SO(6) \simeq SU(4)$ symmetry on the compact dimensions and the CFT side is $\mathcal{N} = 4$ SYM. Here the $SU(4)$ R-symmetry ("rotations" of the 4 supersymmetries) maps to the $SO(6)$.

1. AdS_2/SYK duality.
2. $AdS_4 \times S^7$ in M theory is dual to $\mathcal{N} = 6$ $3D$ Chern–Simons theory.

Definition

$\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills (SYM) with Yang-Mills coupling constant g_{YM} is dynamically equivalent to type IIB superstring theory with string length $l_{\rm s} = \sqrt{\alpha'}$ and coupling constant $g_{\rm s}$ on $AdS_5 \times S^5$ with radius of curvature R and N units of $F_{(5)}$ flux on S^5. There are 2 free parameters on each side ($\frac{R}{\sqrt{\alpha'}}$, $g_{\rm s}$ and N,$g_{\rm YM}$)

$$\lambda = g_{\rm YM}^2 N \quad \text{(effective, or 't Hooft coupling)}$$

Main dictionary

$$g_{\rm YM}^2 = 4\pi g_{\rm s} \quad \text{and} \quad \lambda = g_{\rm YM}^2 N = R^4/\alpha'^2.$$

The above definition is the **strongest form** of AdS/CFT.

Different forms

Strong form: It is valid only for $N \to \infty$ and λ fixed but arbitrary. This implies that $g_s \to 0$ so we get back **classical string theory**. Here we can directly use Einstein's general relativity.

Weak form: It is valid only for $N \to \infty$ and $\lambda >> 1$. This implies that (apart from $g_s \to 0$) $\alpha' \to 0$ so we get back **classical supergravity** (in that limit string length becomes zero and we get back point particles).

Gauge/Gravity duality: This is even stronger that the strongest form of AdS/CFT. It says that we can always find a gauge theory for any arbitrary string theory.

Weak and strong forms are used for the **verification** or trying to **understand strongly coupled field theories**. CFTs because of the high symmetry can sometimes be solved even in the limit of strong coupling. The strongest form is used for **understanding quantum gravity**. Gauge/Gravity Duality is very hard because verification is hard in generic QFTs whose non-perturbative nature is hard to know.

Dictionary

1. Bulk fields ($\phi_{(i)}(x_0, \vec{x}, t)$, i.e. after KK reduction) are related to CFT primary operators ($\mathcal{O}_{(i)}$) as

$$\phi_{(i)}(x_0, \vec{x}, t) = \lim_{x_0 \to 0} x_0^{\Delta_{(i)}} \phi_{0(i)}(\vec{x}, t)$$

   where $\phi_{0(i)}(\vec{x}, t)$ is the source of $\mathcal{O}_{(i)}$. $\Delta_{(i)}$ is the scaling dimension of $\mathcal{O}_{(i)}$.
2. $g_{\mu\nu}$ (bulk) $\Leftrightarrow$ $T_{\mu\nu}$ (boundary).
3. Gauge fields (bulk) $\Leftrightarrow$ conserved currents (boundary).
4. Branes (bulk) $\Leftrightarrow$ solitons (boundary).
5. Radial direction (bulk) $\Leftrightarrow$ energy scale (boundary).
6. Extremal surfaces (bulk) $\Leftrightarrow$ von Neumann entropy (boundary).

The dictionary is **incomplete** and researchers are continuously finding new relations.

Witten prescription

In Euclidean space for CFT

$$Z_{\mathcal{O}}\left[\phi_0\right]=\int \mathcal{D}[SYM \text{ fields }] \exp \left(-S_{\mathcal{N}=4 SYM}+\int d^4 x \mathcal{O}(x) \phi_0(x)\right),$$

$$\left\langle\mathcal{O}\left(x_1\right) \ldots \mathcal{O}\left(x_n\right)\right\rangle=\left.\frac{\delta^n}{\delta \phi_0\left(x_1\right) \ldots \delta \phi_0\left(x_n\right)} Z_{\mathcal{O}}\left[\phi_0\right]\right|_{\phi_0=0}$$

Witten prescription is the natural prescription that both boundary and bulk have same partition function

$$Z_{\mathcal{O}}\left[\phi_0\right]_{\mathrm{CFT}}=Z_\phi\left[\phi_0\right]_{\text{string }} .$$

If we go to the **weak form limit** of AdS/CFT $\Rightarrow$ classical supergravity $\Rightarrow$ no need to calculate the path integrals, **the classical path alone is sufficient**

$$Z\left[\phi_0\right]=\exp \left[-S_{\text{sugra }}\left[\phi\left[\phi_0\right]\right]\right],$$

Express classical solution as a function of the boundary source, $\phi\left[\phi_0\right]$, and replaces it in S_{sugra}. .

Bulk to boundary propagator

$$\Box_{\vec{x},x_0} K_B\left(\vec{x}, x_0; \vec{x}'\right) = \delta^4\left(\vec{x} - \vec{x}'\right),$$
$$\Rightarrow \phi\left(\vec{x}, x_0\right) = \int d^4\vec{x}' K_B\left(\vec{x}, x_0; \vec{x}'\right)\phi_0\left(\vec{x}'\right),$$

and we can substitute this in $S_{\text{sugra}}\left[\phi\right]$.
For a scalar field, the bulk to boundary propagator in AdS_{d+1} is

$$K_{B,\Delta}\left(\vec{x}, x_0; \vec{x}'\right) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma\left(\Delta - \frac{d}{2}\right)}\left[\frac{x_0}{x_0^2 + \left(\vec{x} - \vec{x}'\right)^2}\right]^{\Delta} \equiv C_d\left[\frac{x_0}{x_0^2 + \left(\vec{x} - \vec{x}'\right)^2}\right]^{\Delta}$$

On the boundary, $x_0 \to 0$, this must satisfy

$$K_{B,\Delta}\left(\vec{x}, x_0; \vec{x}'\right) \to x_0^{\Delta}\delta\left(\vec{x} - \vec{x}'\right),$$

so that $\phi\left[\phi_0\right] \to x_0^{\Delta}\phi_0$.

2 point functions

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle = \frac{\delta^2}{\delta\phi_0(x_1)\,\delta\phi_0(x_2)} \exp\left[-S_{\text{sugra}}\;[\phi[\phi_0]]\right]\Big|_{\phi_0=0}$$

We know that

$$S_{\text{sugra}}\;[\phi[\phi_0]] = \frac{1}{2}\int \left(d^5x\sqrt{g}\right)\int d^4\vec{x}' \times \int d^4\vec{y}'\,\partial_\mu K_B\left(\vec{x},x_0;\vec{x}'\right)\phi_0\left(\vec{x}'\right)\partial^\mu K_B\left(\vec{x},x_0;\vec{y}'\right)\phi_0\left(\vec{y}'\right) + O\left(\phi_0^3\right)$$

Now using the fact that first two derivatives of $S_{\text{sugra}}\;[\phi[\phi_0]]$ are zero (since it starts with a quadratic term) and using the chain rule we get

2 point functions

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle = \frac{\delta}{\delta\phi_0[x_1]}\left(-\frac{\delta S_{\text{sugra}}}{\delta\phi_0[x_2]}e^{-S_{\text{sugra}}}\right)\Bigg|_{\phi_0=0} = -\frac{\delta^2 S_{\text{sugra}}\;[\phi[\phi_0]]}{\delta\phi_0(x_1)\,\delta\phi_0(x_2)}\Bigg|$$

$$= -\frac{\delta^2}{\delta\phi_0[x_1]\,\delta\phi_0[x_2]}\frac{1}{2}\int d^5x\sqrt{g}\int d^4\vec{x}'\int d^4\vec{y}'\,\partial_{\mu_{\vec{x}},x_0}K_B(\vec{x},x_0;\vec{x}')\,\phi_0(\vec{x}')$$

$$\times\,\partial^{\mu}_{\vec{x},x_0}K_B(\vec{x},x_0;\vec{y}')\,\phi_0(\vec{y}')$$

$$= -\int d^5x\sqrt{g}\,\partial_{\mu_{\vec{x}},x_0}K_B(\vec{x},x_0;\vec{x}_1)\,\partial^{\mu}_{\vec{x},x_0}K_B(\vec{x},x_0;\vec{x}_2)\,.$$

Now we can substitute the K_B (for 2 point functions there is a simpler way) and find that

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle = -\frac{C_d\,d}{|\vec{x}-\vec{x}'|^{2d}}$$

Fine-grained vs coarse-grained entropy

von Neuman entropy = fine-grained entropy, measures our ignorance about the precise quantum state of the system. Fine-grained entropy is invariant under unitary time evolution $\rho \to U\rho U^{-1}$. So, for a total system it will be constant in time.

Coarse-grained entropy

Definition: For a density matrix ρ we do not measure all observables, and instead we only measure a subset of observables A_i. Then if we consider all possible density matrices $\tilde{\rho}$ which give the same result as our system for the observables that we are tracking, $\text{Tr}\left[\tilde{\rho}A_i\right] = \text{Tr}\left[\rho A_i\right]$. Then we compute the von Neumann entropy $S(\tilde{\rho})$. Finally we maximize this over all possible choices of $\tilde{\rho}$.
Familiar from thermodynamics. Always increases. The thermodynamic entropy is obtained by maximizing the von Neumann entropy among all states with that approximate energy and volume. Coarse-grained entropy defines an **arrow of time** the second law of thermodynamics. Because of coarse-grained entropy's definition, the **fine-grained entropy cannot be bigger than the coarse-grained entropy**.

Bekenstein-Hawking entropy

$$S_{\mathrm{BH}} = \frac{k_{\mathrm{B}} A}{4\ell_{\mathrm{P}}^2} = \frac{A}{4}$$

1. Hawking theorem & Bekenstein entropy: Classically it always **increases**. Even when black hole energy or mass decreases like in Penrose process.
2. Bekenstein-Hawking entropy & Hawking radiation: But once we consider QFT in curved spacetime (**semiclassical** regime) S_{BH} might decrease as long as S_{gen} **increases**. S_{out} is the von Neuman entropy of outside fields.

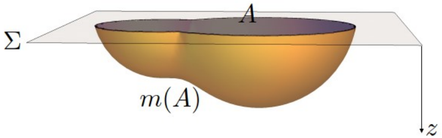
$$S_{\mathrm{BH}} = \frac{A}{4} + S_{out}$$

Both are **coarse-grained**.

Ryu–Takayanagi formula

Same 2 formualas repeat for **fine-grained** cases.
Ryu–Takayanagi prescription: For a holographic spacetime, let σ be a bulk Cauchy slice and let A be a sub-region of its conformal boundary Σ. We assume for the moment that the full boundary is in a pure state, we need to look for the minimal surface $m(A)$ in the bulk that separates A and A^c , as shown in 3.1, in other words that divides σ into a region $r(A)$ bounded by A and another one $r\left(A^c\right)$ bounded by A^c, and assert that the area of $m(A)$ gives the entropy of A :

$$S(A)=\frac{1}{4 G_{\mathrm{N}}} \operatorname{area}(m(A)).$$



Example: 2D CFT interval

The most commonly used entanglement formula. Without RT formula the proof is lengthy and complicated using the replica trick.

$$S(A) = \frac{c}{3} \ln \frac{L}{\epsilon},$$

"Area" here is geodesic length in AdS_3.

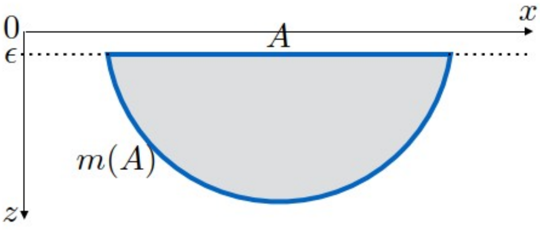


Figure: The inverse radial coordinate z has a UV cut off ϵ.

What if bulk is not classical?

Ryu–Takayanagi: when bulk = classical (i.e. $N \to \infty$)
Quantum Extremal Surfaces by Engelhardt, Wall:

$$S_{gen}(A) = \frac{1}{4G_{\mathrm{N}}}\,\mathrm{area}(QES(A)) + S_{in}(QES(A)).$$

1. The QES minimizes the generalised entropy.
2. S_{in} is the von Neumann entropy of fields inside.
3. This might look like a semiclassical formula but actually it is valid even in full quantum gravity theory if we update the area to a quantum operator.

BH information paradox

Hawking's version : Once the black hole is evaporated the radiation that remains is thermal and therefore doesn't contain the initial information, this is at odds with quantum physics.
AMPS firewall or Mathur's version: Paradox within semiclassical regime. Hawking calculated **fine grained entropy of radiation** (since the total state we take is pure this is also the fine grained entropy of black hole) but it is not less than the coarse grained entropy of black hole (Bekenstein-Hawking entropy). **So the paradox is that after page time coarse grained entropy of black hole is actually less than the fine grained entropy of black hole.**
We need to give up 1 of these: Unitarity, equivalence principle, locality. People now gave up on locality.

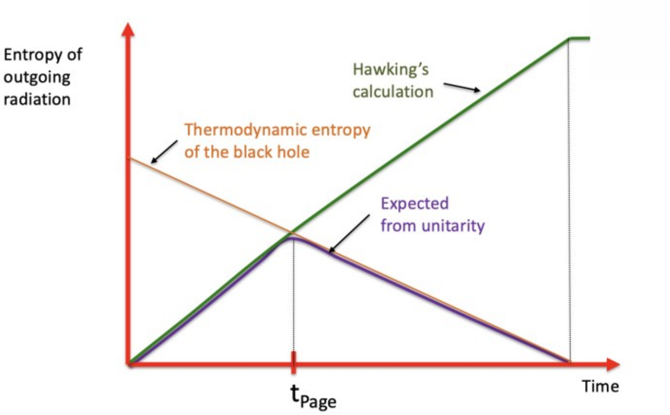


Figure: Schematic behavior of the different entropies. Only the orange one is a coarse gained entropy, the remaining 2 are fine grained.

black hole degrees of freedom + radiation = a pure state, then the **fine-grained entropy** of the black hole should be equal to that of the radiation $S_{\text{black hole}} = S_{\text{rad}}$. Their coarse grained entropies need not be same. But this fine-grained entropy of the black hole should be less than the Bekenstein-Hawking or thermodynamic entropy of the black hole, $S_{\text{black hole}} \leq S_{\text{Bekenstein-Hawking}} = S_{\text{coarse-grained}}$. But after Page time $S_{\text{Bekenstein-Hawking}} < S_{\text{black hole}} = S_{\text{rad}}$.

Islands

Island prescription: Island is a region bounded by a quantum extremal surface. The island is non-locally connected to the outside region. The correct formula for the entropy of the radiation is given by

$$S_{\text{Rad}} = \min_{X}\left\{\text{ext}_X\left[\frac{\text{Area}(X)}{4G_N} + S_{\text{semi-cl}}\left[\Sigma_{\text{Rad}} \cup \Sigma_{\text{Island}}\right]\right]\right\}$$

'Rad' appears both on the LHS and RHS, but the LHS is the full fine-grained entropy of radiation, as computed using the gravitational fine-grained entropy formula. This is supposed to be the entropy for the full exact quantum state of the radiation. RHS: state of radiation in the **semiclassical description** (approximate description). This is a different state than the full exact state of the radiation. Luckily in order to apply the formula we do not need to know the exact state of the radiation. The island formula works for anything not just the black hole radiation.

This island formula was initially derived within holography but later it was derived using just semi classical path integrals and the replica method. The replica method calculation involves wormholes called replica wormholes.

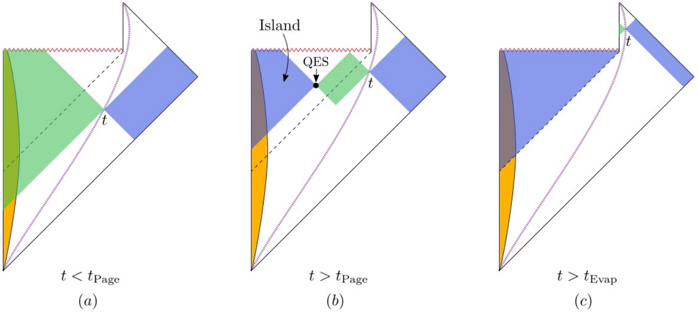


Figure: Island appears around the Page time. At late times the island completely occupies the black hole interior.

Now we have two solutions 1) Trivial island solution (i.e. no islands at all) and 2) Non trivial island solution. We need to take the minimum of these at each point in time to calculate fine grained entropy as shown in Fig 3.5. There is a **phase transition** at the Page time.

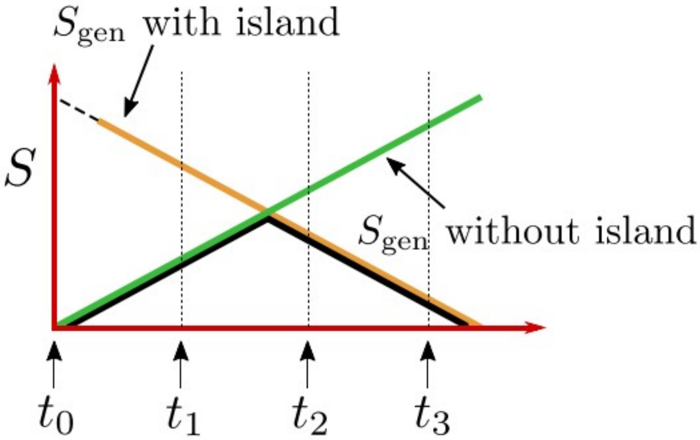
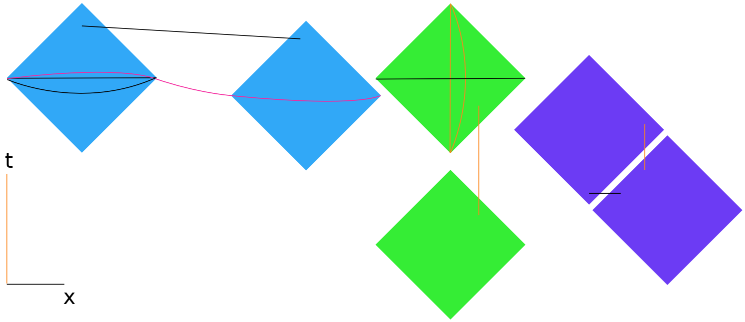


Figure: Taking minima at each point gives the Page curve.

Timelike entanglement

Violation of the CHSH inequality $\implies$ spacelike entanglement.
Violation of the Leggett–Garg inequality $\implies$ timelike entanglement.
Experimentally verified.



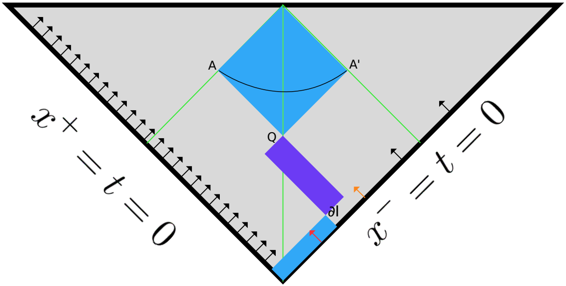


Figure: The island is the lower blue region. The left horizon is evaporating but the right horizon is growing in this non-equilibrium state. Note that the red particle will go inside the left horizon before the orange particle. Because of that, we can get information about the red particle before the orange particle using Hawking radiation. This is reflected in the island covering the red particle even before it covers the orange particle. The timelike entanglement entropy for the union of CFT in blue regions is equal to the value for the purple region.

JT gravity

Einstein's general relativity is topological in $1+1$ dimensions. But, in JT gravity, because of the dilaton, we can see interesting dynamics. The action of de Sitter JT gravity with conformal matter is

$$I = \frac{\Phi_0}{2\pi}\left(\int \mathrm{d}^2x\sqrt{-g}R - 2\int \mathrm{d}x\sqrt{|h|}K\right) + \frac{1}{2\pi}\left(\int \mathrm{d}^2x\sqrt{-g}\Phi\left(R - \frac{2}{\ell^2}\right) - 2\int \mathrm{d}x\sqrt{|h|}\Phi K\right) + I_{\mathrm{CFT}}.$$

Euler–Lagrange equations for the dilaton and the metric, respectively, are

$$R - 2/\ell^2 = 0,$$
$$\Phi g_{\mu\nu} - \ell^2\nabla_\mu\nabla_\nu\Phi + \ell^2 g_{\mu\nu}\Box\Phi = \pi\ell^2\langle T_{\mu\nu}\rangle .$$

Here $\langle T_{\mu\nu}\rangle$ is the expectation value of the matter quantum fields. Notice that **all the backreaction** caused by the introduction of the matter is contained in the dilaton field and that the metric remains unchanged.